

ON THE THREE-DIMENSIONAL MAGNETOHYDRODYNAMICS SYSTEM IN SCALING-INVARIANT SPACES

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ABSTRACT. We study the criterion for the velocity and magnetic vector fields that solve the three-dimensional magnetohydrodynamics system, given any initial data sufficiently smooth, to experience a finite-time blowup. Following the work of [12] and making use of the structure of the system, we obtain a criterion that is imposed on the magnetic vector field and only one of the three components of the velocity vector field, both in scaling-invariant spaces.

Keywords: Navier-Stokes equations, Magnetohydrodynamics system, global regularity, anisotropic Littlewood-Paley theory.

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1. INTRODUCTION AND STATEMENT OF RESULTS

We study the following magnetohydrodynamics system in \mathbb{R}^3 :

$$\frac{du}{dt} + (u \cdot \nabla)u - (b \cdot \nabla)b + \nabla\pi = \nu\Delta u, \quad (1a)$$

$$\frac{db}{dt} + (u \cdot \nabla)b - (b \cdot \nabla)u = \eta\Delta b, \quad (1b)$$

$$\nabla \cdot u = \nabla \cdot b = 0, \quad (u, b)(x, 0) = (u_0, b_0)(x), \quad (1c)$$

where $u, b : \mathbb{R}^3 \times \mathbb{R}^+ \mapsto \mathbb{R}^3$ represent the velocity and magnetic vector fields respectively while $\pi : \mathbb{R}^3 \times \mathbb{R}^+ \mapsto \mathbb{R}$ the scalar pressure field, $\nu \geq 0$ the viscosity and $\eta \geq 0$ the magnetic diffusivity. Without loss of generality, hereafter we assume $\nu = \eta = 1$ and also write ∂_t for $\frac{d}{dt}$ and ∂_i for $\frac{d}{dx_i}$, $i = 1, 2, 3$, $x = (x_1, x_2, x_3)$. Let us also set for any three-dimensional vector field $f = (f^1, f^2, f^3)$, $f^h \triangleq (f^1, f^2, 0)$,

$$\Omega \triangleq \nabla \times u, \quad j \triangleq \nabla \times b, \quad \omega \triangleq \Omega \cdot e^3, \quad d \triangleq j \cdot e^3 \quad \text{where } e^3 \triangleq (0, 0, 1).$$

When $b \equiv 0$, (1a)-(1c) recovers the Navier-Stokes equations (NSE), for which the question of whether a smooth local solution can preserve its regularity for all time remains unknown. The analogous problem for the MHD system (1a)-(1c) remains just as difficult, if not more. One of the sources of the difficulty of the global regularity issue of the MHD system (1a)-(1c) may be traced back to the rescaling and its known bounded quantities. It can be shown that if $(u, b)(x, t)$ solves the system (1a)-(1c), then so does $(u_\lambda, b_\lambda)(x, t) \triangleq \lambda(u, b)(\lambda x, \lambda^2 t)$ while $\|u_\lambda(x, t)\|_{L^2}^2 + \|b_\lambda(x, t)\|_{L^2}^2 = \lambda^{-1}(\|u(x, \lambda^2 t)\|_{L^2}^2 + \|b(x, \lambda^2 t)\|_{L^2}^2)$.

In two-dimensional case, both the NSE and the MHD system, if $\nu, \eta > 0$, admit a unique global smooth solution starting from any data sufficiently smooth (cf.

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[22, 24]). Due to the difficulty in the three-dimensional case, much effort has been devoted to provide regularity and blow-up criterion some of which we review now.

In [25], the author initiated important research direction of regularity criterion which led to, along with others such as [13], that if a weak solution u of the three-dimensional NSE with $\nu > 0$ satisfies

$$u \in L^r(0, T; L^p(\mathbb{R}^3)), \quad \frac{3}{p} + \frac{2}{r} \leq 1, p \in [3, \infty],$$

then u is smooth. Among many other results, in [3] it was shown that if u solves the NSE with $\nu > 0$ and

$$\nabla u \in L^r(0, T; L^p(\mathbb{R}^3)), \quad \frac{3}{p} + \frac{2}{r} \leq 2, \quad 1 < r \leq 3,$$

then u is a regular solution (cf. also [2, 15]). We emphasize that the norm $\|\cdot\|_{L_T^r L_x^p}$ and $\|\cdot\|_{L_T^r \dot{W}_x^{1,p}}$ are both invariant under the scalings of the solutions to the NSE and the MHD system precisely when $\frac{3}{p} + \frac{2}{r} = 1$, $\frac{3}{p} + \frac{2}{r} = 2$ respectively. For the MHD system, e.g. the author in [26] showed that if $\nabla u, \nabla b \in L^4(0, T; L^2(\mathbb{R}^3))$, then no singularity occurs in $[0, T]$. Moreover, the work in [6] in particular showed that if $[0, T^*)$ is the maximal interval of existence of smooth solution and $T^* < \infty$, then

$$\int_0^{T^*} \|\Omega\|_{L^\infty} + \|j\|_{L^\infty} d\tau = \infty.$$

We note that the authors in [16, 36] independently realized that in particular the criterion for the solution to the MHD system may be reduced to just u , dropping condition on b completely (see also [17, 27]). Let us also state results that are directly related to our work. In [14] the authors showed that given $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, there exists a maximal interval $[0, T^*)$ on which a unique solution u to the NSE exists. Analogous results with $(u_0, b_0) \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ may be found in [23, 35].

We now survey some component reduction results of such conditions. The authors in [21] showed that if u solves the NSE with $\nu > 0$ and

$$u^3 \in L^r(0, T; L^p(\mathbb{R}^3)), \quad \frac{3}{p} + \frac{2}{r} \leq \frac{5}{8}, \quad r \in [\frac{54}{23}, \frac{18}{5}],$$

$$\text{or } \nabla u^3 \in L^r(0, T; L^p(\mathbb{R}^3)), \quad \frac{3}{p} + \frac{2}{r} \leq \frac{11}{6}, \quad r \in [\frac{24}{5}, \infty],$$

then the solution is regular (see also [7, 37] for similar results on $u^3, \nabla u^3$). This result was successfully extended to the MHD system as the authors in [20] showed that if u solves (1a)-(1c) with $\nu, \eta > 0$ and

$$u^3, b \in L^r(0, T; L^p(\mathbb{R}^3)), \quad \frac{3}{p} + \frac{2}{r} \leq \frac{3}{4} + \frac{1}{2p}, \quad p > \frac{10}{3}, \quad (2)$$

then the solution pair (u, b) remains smooth for all time. In [32], the author reduced this constraint on u^3, b to u^3, b^1, b^2 in special cases without worsening the upper bound of $\frac{3}{4} + \frac{1}{2p}$ making use of the structure of (1b). We note however that the upper bound of $\frac{3}{4} + \frac{1}{2p}$ does not allow the norm to be scaling-invariant. For more interesting component reduction results of such criterion, we refer to e.g. [8, 9, 30, 31]. In particular, we point out that the author in [33] obtained the following

regularity criterion for the solution to the three-dimensional MHD system:

$$\begin{cases} u^3 \in L^{r_1}(0, T; L^{p_1}(\mathbb{R}^3)), & \frac{3}{p_1} + \frac{2}{r_1} \leq \frac{1}{3} + \frac{1}{2p_1}, \quad \frac{15}{2} < p_1, \\ d \in L^{r_2}(0, T; L^{p_2}(\mathbb{R}^3)), & \frac{3}{p_2} + \frac{2}{r_2} \leq 2, \quad \frac{3}{2} < p_2. \end{cases} \quad (3)$$

We remark that the upper bound of p_2, r_2 for d allows the scaling-invariant case.

We now motivate our study. In [12], the authors succeeded in showing that given an initial data for the NSE in $\dot{W}^{1, \frac{3}{2}}(\mathbb{R}^3)$, if blow-up occurs at $T^* < \infty$, then $u_3 \notin L^p(0, T^*; \dot{H}^{\frac{1}{2} + \frac{2}{p}})$, $p \in (4, 6)$. The purpose of this manuscript is to extend this result to the MHD system (1a)-(1c). We emphasize that because the proof in [12] required taking a curl of the NSE and studying its vorticity formulation carefully, such a generalization to the MHD system is non-trivial. A well-known example for this type of difficulty is that although in two-dimensional case, the author in [34] showed that the solution to the Euler equations admits a unique global smooth solution, it remains unknown if such a result may be extended to the two-dimensional MHD system even with full magnetic diffusion (see [10] and references found therein). Similarly, although in [11], the authors obtained a two-vorticity component regularity criterion for the three-dimensional NSE making use of the Biot-Savart law in a special way, to the best of the author's knowledge, an analogous criterion, e.g. in terms of two-vorticity components, does not exist for the MHD system. The difficulty in extending these results is due to the current density formulation upon taking a curl on (1b) (in particular $M(u, b)$ in (19)). This difficulty appears even in two-dimensional case, e.g. [18] in which the author elaborates why the current density formulation is not as simple as that of vorticity.

We let $\Omega_0(x) = \Omega(x, 0)$, $j_0(x) = j(x, 0)$ and denote the following norm which is invariant under the scaling of the solution to the MHD system

$$\|f\|_{SC_{p, p_1, p_2}} \triangleq \|f\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}}^p + \|f\|_{L^{p_1}}^{r_1} + \|\nabla f\|_{L^{p_2}}^{r_2}, \quad \frac{3}{p_1} + \frac{2}{r_1} = 1, \quad \frac{3}{p_2} + \frac{2}{r_2} = 2.$$

We remark that $\int_0^T \|b_\lambda(x, \tau)\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}}^p d\tau = \int_0^{\lambda^2 T} \|b(x, \tau)\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}}^p d\tau$. Our results read

Theorem 1.1. *Suppose $\Omega_0, j_0 \in L^{\frac{3}{2}}(\mathbb{R}^3)$. Then $\exists!$ solution pair (u, b) to the MHD system (1a)-(1c) such that $u, b \in C(0, T^*; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)) \cap L_{loc}^2(0, T^*; \dot{H}^{\frac{3}{2}}(\mathbb{R}^3))$ and*

$$\begin{aligned} & \sup_{t \in [0, T]} (\|\Omega\|_{L^{\frac{3}{2}}}^{\frac{3}{2}} + \|j\|_{L^{\frac{3}{2}}}^{\frac{3}{2}})(t) + \int_0^T |\nabla(\Omega + j)|^2 |\Omega + j|^{-\frac{1}{2}} + |\nabla(\Omega - j)|^2 |\Omega - j|^{-\frac{1}{2}} d\tau \\ & \leq c(1 + \|\Omega_0\|_{L^{\frac{3}{2}}}^{\frac{3}{2}} + \|j_0\|_{L^{\frac{3}{2}}}^{\frac{3}{2}}) \exp \left(\int_0^T \|u\|_{\dot{H}^{\frac{3}{2}}}^2 + \|b\|_{\dot{H}^{\frac{3}{2}}}^2 d\tau \right) < \infty \end{aligned}$$

$\forall T < T^*$. Moreover, let $p \in (4, 6)$, $p_1 > 9$, $p_2 > \frac{9}{2}$. If $T^* < \infty$, then

$$\int_0^{T^*} \|u^3\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}}^p + \|b\|_{SC_{p, p_1, p_2}} d\tau = \infty. \quad (4)$$

Remark 1.1. (1) In comparison with (2) and (3), the conditions in (4) is at the scaling-invariant level.

(2) Taking $b \equiv 0$ recovers the result in [12].

(3) The difficulty in the estimate of the third component of the curl formulation in contrast to the case of the NSE is in particular $M(u, b)$ in (19). The difficulty in the estimate of the third component of the velocity equation is

the pressure term which now involves the quadratic of the magnetic field (see (41a)). We had to take advantage of the structure of the MHD system, in particular make cancellations such as in (21)-(23) (see also (43a)-(43e)) and $(V_1 + VI_1)_1$ and $(V_1 + VI_1)_4$ in (82).

In the Preliminaries we set up notations and state key lemmas. Thereafter, we prove the second statement of Theorem 1.1, namely (4). Because local existence theory is classical, we sketch it in the Appendix for completeness.

2. PRELIMINARIES

We write $A \lesssim_{a,b} B$ when there exists a constant $c \geq 0$ of significant dependence only on a, b such that $A \leq cB$, similarly $A \approx_{a,b} B$ if $A = cB$. For simplicity, we denote $\int = \int_{\mathbb{R}^3}$ and omit dx when no confusion arises. We also denote

$$\nabla_h^\perp \triangleq (-\partial_2, \partial_1, 0), \quad \Delta_h \triangleq \sum_{i=1}^2 \partial_i^2,$$

with which we may write down the key identity to be used frequently in our proof, namely $\forall f = (f^h, f^3)$ such that $\nabla \cdot f = 0$,

$$f^h = f_{\text{curl}}^h + f_{\text{div}}^h \quad \text{where} \quad f_{\text{curl}}^h \triangleq \nabla_h^\perp \Delta_h^{-1} (\nabla \times f) \cdot e^3, \quad f_{\text{div}}^h \triangleq -\nabla_h \Delta_h^{-1} \partial_3 f^3. \quad (5)$$

We also denote for any scalar function f^i , $f_\alpha^i \triangleq \frac{f^i}{|f^i|} |f^i|^\alpha$, $\alpha \in \mathbb{R}^+$.

We recall the anisotropic Lebesgue spaces reminding ourselves that its order matters, i.e. $\|f(\cdot, x_2)\|_{L^p(X_1, \mu_1)} \|f(x_1, \cdot)\|_{L^q(X_2, \mu_2)} \leq \|f(x_1, \cdot)\|_{L^q(X_2, \mu_2)} \|f(\cdot, x_2)\|_{L^p(X_1, \mu_1)}$ for any two measure spaces $(X_1, \mu_1), (X_2, \mu_2)$ with $1 \leq p \leq q \leq \infty$ (cf. [4]). Let us denote by \mathcal{S} the Schwartz space and \mathcal{S}' its dual. We continue to use the following definitions of anisotropic Sobolev spaces from [12] (see also [19, 29]).

Definition 2.1. For $s, s' \in \mathbb{R}$, $\dot{H}^{s, s'}$ denotes the space of $f \in \mathcal{S}'$ such that

$$\|f\|_{\dot{H}^{s, s'}}^2 \triangleq \int_{\mathbb{R}^3} |\xi_h|^{2s} |\xi_3|^{2s'} |\hat{f}(\xi)|^2 d\xi < \infty, \quad \xi_h = (\xi_1, \xi_2, 0).$$

Moreover, for $\theta \in (0, \frac{1}{2})$, we denote $\mathcal{H}_\theta \triangleq \dot{H}^{-\frac{1}{2} + \theta, -\theta}$.

We recall the Littlewood-Paley decomposition; with χ, ϕ smooth functions such that

$$\begin{aligned} \text{supp } \phi &\subset \{\xi \in \mathbb{R} : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}, & \sum_{j \in \mathbb{Z}} \phi(2^{-j} \xi) &= 1, \\ \text{supp } \chi &\subset \{\xi \in \mathbb{R} : |\xi| \leq \frac{4}{3}\}, & \chi(\xi) + \sum_{j \geq 0} \phi(2^{-j} \xi) &= 1, \end{aligned}$$

we denote the Littlewood-Paley operators, classical and anisotropic,

$$\begin{aligned} \dot{\Delta}_j f &\triangleq \mathcal{F}^{-1}(\phi(2^{-j} |\xi|) \hat{f}), & \dot{S}_j f &\triangleq \mathcal{F}^{-1}(\chi(2^{-j} |\xi|) \hat{f}), \\ \dot{\Delta}_k^h f &\triangleq \mathcal{F}^{-1}(\phi(2^{-k} |\xi_h|) \hat{f}), & \dot{S}_k^h f &\triangleq \mathcal{F}^{-1}(\chi(2^{-k} |\xi_h|) \hat{f}), \\ \dot{\Delta}_l^v f &\triangleq \mathcal{F}^{-1}(\phi(2^{-l} |\xi_3|) \hat{f}), & \dot{S}_l^v f &\triangleq \mathcal{F}^{-1}(\chi(2^{-l} |\xi_3|) \hat{f}). \end{aligned}$$

We define \mathcal{S}'_h to be the subspace of \mathcal{S}' such that $\lim_{j \rightarrow -\infty} \|\dot{S}_j f\|_{L^\infty} = 0 \quad \forall f \in \mathcal{S}'_h$.

Definition 2.2. For $p, q \in [1, \infty]$, $s \in \mathbb{R}$, $s < \frac{3}{p}(s = \frac{3}{p}$ if $q = 1$), we define the Besov spaces $\dot{B}_{p,q}^s(\mathbb{R}^3) \triangleq \{f \in \mathcal{S}'_h : \|f\|_{\dot{B}_{p,q}^s} < \infty\}$ where

$$\|f\|_{\dot{B}_{p,q}^s} \triangleq \left\| (2^{js} \|\dot{\Delta}_j f\|_{L^p})_j \right\|_{l^q(\mathbb{Z})}.$$

Moreover, for $p \in (1, \infty)$, we shall use the notations $\mathcal{B}_p \triangleq \dot{B}_{\infty,\infty}^{-2+\frac{2}{p}}$.

We define the anisotropic Besov spaces $(\dot{B}_{p,q_1}^{s_1})_h(\dot{B}_{p,q_2}^{s_2})_v$ as the space of distributions in \mathcal{S}'_h endowed with its norm of

$$\|f\|_{(\dot{B}_{p,q_1}^{s_1})_h(\dot{B}_{p,q_2}^{s_2})_v} \triangleq \left(\sum_{k \in \mathbb{Z}} 2^{q_1 k s_1} \left(\sum_{l \in \mathbb{Z}} 2^{q_2 l s_2} \|\dot{\Delta}_k^h \dot{\Delta}_l^v f\|_{L^p}^{q_2} \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}}.$$

It is well-known that $\dot{B}_{2,2}^s = \dot{H}^s$ (cf. [5]). Moreover, $(\dot{B}_{p,q_1}^{s_1})_h(\dot{B}_{p,q_2}^{s_2})_v|_{p=q_1=q_2=2} = \dot{H}^{s_1, s_2}$. We recall the important Bony's para-product decomposition:

$$fg = T(f, g) + T(g, f) + R(f, g) \quad (6)$$

$$\text{where } T(f, g) \triangleq \sum_{j \in \mathbb{Z}} \dot{S}_{j-1} f \dot{\Delta}_j g, \quad R(f, g) \triangleq \sum_{j \in \mathbb{Z}} \dot{\Delta}_j f \tilde{\Delta}_j g, \quad \text{with } \tilde{\Delta}_j \triangleq \sum_{l=j-1}^{j+1} \dot{\Delta}_l.$$

We also recall the useful anisotropic Bernstein's inequalities:

Lemma 2.1. Let \mathcal{B}_h (respectively \mathcal{B}_v) a ball in \mathbb{R}_h^2 (resp. \mathbb{R}_v) and \mathcal{C}_h (resp. \mathcal{C}_v) a ring in \mathbb{R}_h^2 (resp. \mathbb{R}_v). Moreover, let $1 \leq p_2 \leq p_1 \leq \infty$, $1 \leq q_2 \leq q_1 \leq \infty$. Then

$$\begin{aligned} \|\nabla_h^\alpha f\|_{L_h^{p_1}(L_v^{q_1})} &\lesssim 2^{k(|\alpha|+2(\frac{1}{p_2}-\frac{1}{p_1}))} \|f\|_{L_h^{p_2}(L_v^{q_1})}, & \text{if } \text{supp } \hat{f} \subset 2^k \mathcal{B}_h, \\ \|\partial_3^\beta f\|_{L_h^{p_1}(L_v^{q_1})} &\lesssim 2^{l(|\beta|+(\frac{1}{q_2}-\frac{1}{q_1}))} \|f\|_{L_h^{p_1}(L_v^{q_2})}, & \text{if } \text{supp } \hat{f} \subset 2^l \mathcal{B}_v, \\ \|f\|_{L_h^{p_1}(L_v^{q_1})} &\lesssim 2^{-kN} \sup_{|\alpha|=N} \|\nabla_h^\alpha f\|_{L_h^{p_1}(L_v^{q_1})}, & \text{if } \text{supp } \hat{f} \subset 2^k \mathcal{C}_h, \\ \|f\|_{L_h^{p_1}(L_v^{q_1})} &\lesssim 2^{-lN} \|\partial_3^N f\|_{L_h^{p_1}(L_v^{q_1})}, & \text{if } \text{supp } \hat{f} \subset 2^l \mathcal{C}_v. \end{aligned}$$

We also recall in relevance the product law in anisotropic spaces (cf. Lemma 4.5 [12]): for $q \geq 1$, $p_1 \geq p_2 \geq 1$, $\frac{1}{p_1} + \frac{1}{p_2} \leq 1$, $s_1 < \frac{2}{p_1}$, $s_2 < \frac{2}{p_2}$ (resp. $s_1 \leq \frac{2}{p_1}$, $s_2 \leq \frac{2}{p_2}$ if $q = 1$), $s_1 + s_2 > 0$, $\sigma_1 < \frac{1}{p_1}$, $\sigma_2 < \frac{1}{p_2}$ (resp. $\sigma_1 \leq \frac{1}{p_1}$, $\sigma_2 \leq \frac{1}{p_2}$ if $q = 1$), $\sigma_1 + \sigma_2 > 0$,

$$\|fg\|_{(\dot{B}_{p_1,q}^{s_1+s_2-\frac{2}{p_2}})_h(\dot{B}_{p_1,q}^{\sigma_1+\sigma_2-\frac{1}{p_2}})_v} \lesssim \|f\|_{(\dot{B}_{p_1,q}^{s_1})_h(\dot{B}_{p_1,q}^{\sigma_1})_v} \|g\|_{(\dot{B}_{p_2,q}^{s_2})_h(\dot{B}_{p_2,q}^{\sigma_2})_v} \quad (7)$$

(cf. also [19, 29]).

We now recall several results from [12] on which we will rely. Firstly, the proof of Proposition 4.1 in [12] verifies the following inequality:

Lemma 2.2. Let $f = (f^1, f^2, f^3)$ satisfy $\nabla \cdot f = 0$ and $g = (\nabla \times f) \cdot e_3$. Then for $\alpha, \theta \in (0, \frac{1}{2})$,

$$\|f^h\|_{(\dot{B}_{2,1}^1)_h(\dot{B}_{2,1}^{\frac{1}{2}-\alpha})_v} \lesssim \|g\|_{L^2}^{\frac{1}{3}+\alpha} \|\nabla g\|_{L^2}^{1-\alpha} + \|\partial_3 f^3\|_{\mathcal{H}_\theta}^\alpha \|\nabla \partial_3 f^3\|_{\mathcal{H}_\theta}^{1-\alpha}.$$

Lemma 2.3. The following inequalities hold for $f = (f^1, f^2, f^3)$:

$$((2.4) [12]) \quad \|\partial_3 f^3\|_{\mathcal{H}_\theta} \lesssim \|f\|_{\dot{H}^{\frac{1}{2}}} \quad \text{if } \nabla \cdot f = 0, \quad (8)$$

$$(Lemma 3.2, (3.8) [12]) \quad \|\nabla f^i\|_{L^{\frac{3}{2}}} \lesssim \|\nabla f_{\frac{3}{4}}^i\|_{L^2} \|f_{\frac{3}{4}}^i\|_{L^2}^{\frac{1}{3}}, \quad \|\nabla f^i\|_{L^{\frac{9}{5}}} \lesssim \|\nabla f_{\frac{3}{4}}^i\|_{L^2}^{\frac{4}{3}}, \quad (9)$$

$$(Lemma\ 3.2\ [12])\ \|f^i\|_{\dot{H}^s} \lesssim \|f_{\frac{3}{4}}^i\|_{L^2}^{\frac{5}{6}-s} \|\nabla f_{\frac{3}{4}}^i\|_{L^2}^{\frac{1}{2}+s}, \quad s \in [-\frac{1}{2}, \frac{5}{6}], \quad (10)$$

$$(pg.\ 31\ [12])\ \|f\|_{\mathcal{B}_p} \lesssim \|f\|_{L^{\frac{3}{2}}}^{1-\frac{3}{2p}} \|\nabla f\|_{L^{\frac{9}{8}}}^{\frac{3}{2p}}, \quad p > \frac{3}{2}, \quad (11)$$

$$(pg.\ 22\ [12])\ \|f\|_{\dot{H}^{\theta, \frac{1}{2}-\theta-\frac{1}{p}}} \lesssim \|f\|_{\mathcal{H}_\theta}^{\frac{1}{p}} \|\nabla f\|_{\mathcal{H}_\theta}^{1-\frac{1}{p}}, \quad p > 2. \quad (12)$$

Lemma 2.4. (Lemma 5.2 [12]) Let $\theta \in (0, \frac{1}{6})$, $\sigma \in (\frac{3}{4}, 1)$, $s = \frac{3}{2} - \frac{2}{3}\sigma$. Then

$$|\int \partial_h \Delta_h^{-1} f \partial_h g h_{\frac{1}{2}}| \lesssim \|f\|_{L^{\frac{3}{2}}} \|g\|_{\dot{H}^s} \|h_{\frac{3}{4}}\|_{\dot{H}^\sigma}^{\frac{2}{3}}, \quad (13a)$$

$$|\int \partial_h \Delta_h^{-1} f \partial_h g h_{\frac{1}{2}}| \lesssim \|f\|_{\mathcal{H}_\theta} \|g\|_{\dot{H}^s} \|h_{\frac{3}{4}}\|_{\dot{H}^\sigma}^{\frac{2}{3}}. \quad (13b)$$

Lemma 2.5. (Lemma 6.1 [12]) Let A be a bounded Fourier multiplier, f, g, h any scalar-valued functions. Then for p, θ such that $0 < \theta < \frac{1}{2} - \frac{1}{p}$,

$$|(A(D)(fg)|\partial_3 h)_{\mathcal{H}_\theta}| \lesssim \|f\|_{\dot{H}^{\theta, \frac{1}{2}-\theta-\frac{1}{p}}} \|g\|_{\dot{H}^{\theta, \frac{1}{2}-\theta-\frac{1}{p}}} \|h\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}. \quad (14)$$

Moreover, for such θ and p , (pg. 22 [12])

$$\|f\|_{\dot{H}^{\theta, \frac{1}{2}-\theta-\frac{1}{p}}} \lesssim \|f_{\frac{3}{4}}\|_{L^2}^{\frac{p+3}{3p}} \|\nabla f_{\frac{3}{4}}\|_{L^2}^{1-\frac{1}{p}}. \quad (15)$$

We need the following generalized version of Lemma 6.2 from [CZ14] for our purpose; for completeness we sketch its proof:

Lemma 2.6. Let A be a bounded Fourier multiplier and $f = (f^1, f^2, f^3)$, $g = (g^1, g^2, g^3)$, $h = (h^1, h^2, h^3)$. If p, θ satisfy $0 < \theta < \frac{1}{2} - \frac{1}{p}$, $0 < \frac{2}{p}$, then for $l \in \{1, 2\}$,

$$|(A(D)(f^l \partial_l \partial_3 g^3)|\partial_3 h^3)_{\mathcal{H}_\theta}| \lesssim \|f^l\|_{(\dot{B}_{2,1}^1)_h(\dot{B}_{2,1}^{\frac{1}{2}-\frac{2}{p}})_v} \|\nabla \partial_3 g^3\|_{\mathcal{H}_\theta} \|h^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}.$$

In particular, if $p > 4$ and (u, b) solves the MHD system (1a)-(1c), then

$$|(A(D)(u^l \partial_l \partial_3 g^3)|\partial_3 h^3)_{\mathcal{H}_\theta}| \quad (16)$$

$$\lesssim \left(\|\omega_{\frac{3}{4}}\|_{L^2}^{\frac{1}{3}+\frac{2}{p}} \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^{1-\frac{2}{p}} + \|\partial_3 u^3\|_{\mathcal{H}_\theta}^{\frac{2}{p}} \|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta}^{1-\frac{2}{p}} \right) \|\nabla \partial_3 g^3\|_{\mathcal{H}_\theta} \|h^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}},$$

$$|(A(D)(b^l \partial_l \partial_3 g^3)|\partial_3 h^3)_{\mathcal{H}_\theta}| \quad (17)$$

$$\lesssim \left(\|d_{\frac{3}{4}}\|_{L^2}^{\frac{1}{3}+\frac{2}{p}} \|\nabla d_{\frac{3}{4}}\|_{L^2}^{1-\frac{2}{p}} + \|\partial_3 b^3\|_{\mathcal{H}_\theta}^{\frac{2}{p}} \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta}^{1-\frac{2}{p}} \right) \|\nabla \partial_3 g^3\|_{\mathcal{H}_\theta} \|h^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}.$$

Proof. We estimate

$$\begin{aligned} & |(A(D)(f^l \partial_l \partial_3 g^3)|\partial_3 h^3)_{\mathcal{H}_\theta}| \\ & \lesssim \|f^l \partial_l \partial_3 g^3\|_{\dot{H}^{-1+2\theta, \frac{1}{2}-\frac{2}{p}-2\theta}} \|\partial_3 h^3\|_{\dot{H}^{0, -\frac{1}{2}+\frac{2}{p}}} \\ & \lesssim \|f^l\|_{(\dot{B}_{2,1}^1)_h(\dot{B}_{2,1}^{\frac{1}{2}-\frac{2}{p}})_v} \|\partial_l \partial_3 g^3\|_{(\dot{B}_{2,2}^{-1+2\theta})_h(\dot{B}_{2,2}^{\frac{1}{2}-2\theta})_v} \|h^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \\ & \lesssim \|f^l\|_{(\dot{B}_{2,1}^1)_h(\dot{B}_{2,1}^{\frac{1}{2}-\frac{2}{p}})_v} \|\partial_3 g^3\|_{(\dot{B}_{2,2}^{2\theta})_h(\dot{B}_{2,2}^{\frac{1}{2}-2\theta})_v} \|h^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \end{aligned}$$

where we used the fact that $\|h^3\|_{\dot{H}^{0, \frac{1}{2}+\frac{2}{p}}}^2 \lesssim \int |\xi|^{1+\frac{4}{p}} |\hat{h}^3(\xi)|^2 d\xi \approx \|h^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^2$. Moreover, we remark that the second inequality actually cannot be an application of

the product law in anisotropic spaces (7). Nevertheless, it can be justified by a standard technique of anisotropic space estimate (94). Now since

$$\|\partial_3 g^3\|_{(\dot{B}_{2,2}^{2\theta})_h(\dot{B}_{2,2}^{\frac{1}{2}-2\theta})_v}^2 = \int |\xi_h|^{4\theta} |\xi_3|^{1-4\theta} |\widehat{\partial_3 g^3}(\xi)|^2 d\xi \lesssim \|\nabla \partial_3 g^3\|_{\mathcal{H}_\theta}^2$$

which can be verified using that $\theta < \frac{1}{2}$, we obtain

$$|(A(D)(f^l \partial_t \partial_3 g^3)|\partial_3 h^3)_{\mathcal{H}_\theta}| \lesssim \|f^l\|_{(\dot{B}_{2,1}^1)_h(\dot{B}_{2,1}^{\frac{1}{2}-\frac{2}{p}})_v} \|\nabla \partial_3 g^3\|_{\mathcal{H}_\theta}^2 \|h^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}.$$

The particular cases are just consequences Lemma 2.2 with $\alpha = \frac{2}{p}$. This completes the proof of Lemma 2.6. \square

We end this Preliminaries with the following lemma:

Lemma 2.7. (Lemma 4.3 [12]) Let $s > 0, \alpha \in (0, s)$. Then for $f \in \dot{B}_{p,q}^s$,

$$\|f\|_{(\dot{B}_{p,q}^{s-\alpha})_h(\dot{B}_{p,1}^\alpha)_v} \lesssim \|f\|_{\dot{B}_{p,q}^s}.$$

3. THREE PROPOSITIONS

Proposition 3.1. Under the hypothesis of Theorem 1.1, for $\theta \in (0, \frac{1}{6})$ the solution to the MHD system (1a)-(1c) satisfies for any $t < T^*$

$$\begin{aligned} & \frac{2}{3}(\|\omega_{\frac{3}{4}}\|_{L^2}^2 + \|d_{\frac{3}{4}}\|_{L^2}^2)(t) + \frac{5}{9} \int_0^t \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 + \|\nabla d_{\frac{3}{4}}\|_{L^2}^2 d\tau \\ & \lesssim e^{c \int_0^t \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b\|_{SC_{p,p_1,p_2}} d\tau} \\ & \quad \times \left(\frac{2}{3}(\|\omega_{\frac{3}{4}}\|_{L^2}^2 + \|d_{\frac{3}{4}}\|_{L^2}^2)(0) + \left(\int_0^t \|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta}^2 + \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta}^2 d\tau \right)^{\frac{3}{4}} \right). \end{aligned}$$

Remark 3.1. The fact that the MHD system (1a)-(1c) forces a worse bound in terms of $\int_0^t \|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta}^2 + \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta}^2 d\tau$ rather than $\int_0^t \|\partial_3^2 u^3\|_{\mathcal{H}_\theta}^2 d\tau$ in [12] is in particular due to the matrix $M(u, b)$ in (19).

Proof. We take a curl on (1a), (1b) to obtain

$$\partial_t \Omega - \Delta \Omega + (u \cdot \nabla) \Omega - (\Omega \cdot \nabla) u - (b \cdot \nabla) j + (j \cdot \nabla) b = 0 \quad (18a)$$

$$\partial_t j - \Delta j + (u \cdot \nabla) j - (j \cdot \nabla) u - (b \cdot \nabla) \Omega + (\Omega \cdot \nabla) b = 2M(u, b) \quad (18b)$$

where

$$M(u, b) \triangleq \begin{pmatrix} \partial_2 b \cdot \partial_3 u - \partial_3 b \cdot \partial_2 u \\ \partial_3 b \cdot \partial_1 u - \partial_1 b \cdot \partial_3 u \\ \partial_1 b \cdot \partial_2 u - \partial_2 b \cdot \partial_1 u \end{pmatrix}. \quad (19)$$

In particular, the third components of this system reads

$$\partial_t \omega + (u \cdot \nabla) \omega - (b \cdot \nabla) d - \Delta \omega = (\Omega \cdot \nabla) u^3 - (j \cdot \nabla) b^3, \quad (20a)$$

$$\partial_t d + (u \cdot \nabla) d - (b \cdot \nabla) \omega - \Delta d = (j \cdot \nabla) u^3 - (\Omega \cdot \nabla) b^3 + 2[\partial_1 b \cdot \partial_2 u - \partial_2 b \cdot \partial_1 u]. \quad (20b)$$

We make a few important cancellations:

$$(\Omega \cdot \nabla) u^3 = \partial_3 u^3 \omega + \partial_2 u^3 \partial_3 u^1 - \partial_1 u^3 \partial_3 u^2, \quad (21)$$

$$(j \cdot \nabla) b^3 = \partial_3 b^3 d + \partial_2 b^3 \partial_3 b^1 - \partial_1 b^3 \partial_3 b^2. \quad (22)$$

We make cancellations within $2[\partial_1 b \cdot \partial_2 u - \partial_2 b \cdot \partial_1 u]$ as well:

$$\begin{aligned} & (j \cdot \nabla)u^3 - (\Omega \cdot \nabla)b^3 + 2[\partial_1 b \cdot \partial_2 u - \partial_2 b \cdot \partial_1 u] \\ &= \partial_3 u^3 d - \partial_3 b^3 \omega - \partial_3 b^2 \partial_1 u^3 + \partial_3 b^1 \partial_2 u^3 + \partial_3 u^2 \partial_1 b^3 - \partial_3 u^1 \partial_2 b^3 \\ & \quad + 2[\partial_1 b^1 \partial_2 u^1 - \partial_2 b^1 \partial_1 u^1 + \partial_1 b^2 \partial_2 u^2 - \partial_2 b^2 \partial_1 u^2]. \end{aligned} \quad (23)$$

Therefore, we have from (20a)-(20b), (21)-(23),

$$\partial_t \omega + (u \cdot \nabla)\omega - (b \cdot \nabla)d - \Delta \omega = F_1, \quad (24a)$$

$$\partial_t d + (u \cdot \nabla)d - (b \cdot \nabla)\omega - \Delta d = F_2, \quad (24b)$$

$$F_1 \triangleq (\partial_3 u^3 \omega + \partial_2 u^3 \partial_3 u^1 - \partial_1 u^3 \partial_3 u^2) - (\partial_3 b^3 d + \partial_2 b^3 \partial_3 b^1 - \partial_1 b^3 \partial_3 b^2), \quad (24c)$$

$$\begin{aligned} F_2 \triangleq & \partial_3 u^3 d - \partial_3 b^3 \omega - \partial_3 b^2 \partial_1 u^3 + \partial_3 b^1 \partial_2 u^3 + \partial_3 u^2 \partial_1 b^3 - \partial_3 u^1 \partial_2 b^3 \\ & + 2[\partial_1 b^h \cdot \partial_2 u^h - \partial_2 b^h \cdot \partial_1 u^h]. \end{aligned} \quad (24d)$$

Taking $L^{\frac{3}{2}}$ -norm estimate, using divergence-free conditions and that $|\nabla f_{\frac{3}{4}}^i| = \frac{3}{4}|\nabla f^i||f^i|^{-\frac{1}{4}}$, integrating in time we obtain with $\omega(0) = \omega_0, d(0) = d_0$,

$$\begin{aligned} & \frac{2}{3}(\|\omega\|_{L^{\frac{3}{2}}}^{\frac{3}{2}} + \|d\|_{L^{\frac{3}{2}}}^{\frac{3}{2}})(t) + \frac{8}{9} \int_0^t \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 + \|\nabla d_{\frac{3}{4}}\|_{L^2}^2 d\tau \\ &= \frac{2}{3} \left(\|\omega_0\|_{L^{\frac{3}{2}}}^{\frac{3}{2}} + \|d_0\|_{L^{\frac{3}{2}}}^{\frac{3}{2}} \right) + \int_0^t \int (b \cdot \nabla) d \omega_{\frac{1}{2}} + (b \cdot \nabla) \omega d_{\frac{1}{2}} + F_1 \omega_{\frac{1}{2}} + F_2 d_{\frac{1}{2}} d\tau. \end{aligned} \quad (25)$$

We first estimate

$$\begin{aligned} & \int (b \cdot \nabla) d \omega_{\frac{1}{2}} + (b \cdot \nabla) \omega d_{\frac{1}{2}} \\ & \lesssim \int |b| |\nabla d_{\frac{3}{4}}| |d|^{\frac{1}{4}} |\omega|^{\frac{1}{2}} + |b| |\nabla \omega_{\frac{3}{4}}| |\omega|^{\frac{1}{4}} |d|^{\frac{1}{2}} \\ & \lesssim \|b\|_{L^{p_1}} \left(\|\nabla d_{\frac{3}{4}}\|_{L^2} \|d_{\frac{3}{4}}\|_{L^{\frac{2p_1}{p_1-6}}}^{\frac{1}{3}} \|\omega_{\frac{3}{4}}\|_{L^2}^{\frac{2}{3}} + \|\nabla \omega_{\frac{3}{4}}\|_{L^2} \|\omega_{\frac{3}{4}}\|_{L^{\frac{2p_1}{p_1-6}}}^{\frac{1}{3}} \|d_{\frac{3}{4}}\|_{L^2}^{\frac{2}{3}} \right) \\ & \lesssim \|b\|_{L^{p_1}} \left(\|\nabla d_{\frac{3}{4}}\|_{L^2}^{\frac{p_1+3}{p_1}} \|d_{\frac{3}{4}}\|_{L^2}^{\frac{1}{3}(\frac{p_1-9}{p_1})} \|\omega_{\frac{3}{4}}\|_{L^2}^{\frac{2}{3}} + \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^{\frac{p_1+3}{p_1}} \|\omega_{\frac{3}{4}}\|_{L^2}^{\frac{1}{3}(\frac{p_1-9}{p_1})} \|d_{\frac{3}{4}}\|_{L^2}^{\frac{2}{3}} \right) \\ & \leq \frac{1}{18} \left(\|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 + \|\nabla d_{\frac{3}{4}}\|_{L^2}^2 \right) + c \|b\|_{L^{\frac{2p_1}{p_1-3}}}^{\frac{2p_1}{p_1-3}} (\|\omega_{\frac{3}{4}}\|_{L^2}^2 + \|d_{\frac{3}{4}}\|_{L^2}^2) \end{aligned} \quad (26)$$

where we used Hölder's, Gagliardo-Nirenberg and Young's inequalities.

Next, we rearrange terms carefully and estimate differently as follows:

$$\int_0^t \int F_1 \omega_{\frac{1}{2}} + F_2 d_{\frac{1}{2}} d\tau \triangleq \sum_{i=1}^5 I_i \quad (27)$$

where

$$\begin{aligned}
I_1 &= \int_0^t \int \partial_3 u^3 \omega \omega_{\frac{1}{2}} + \partial_3 u^3 d d_{\frac{1}{2}} d\tau, \quad I_2 = - \int_0^t \int \partial_3 b^3 d \omega_{\frac{1}{2}} + \partial_3 b^3 \omega d_{\frac{1}{2}} d\tau, \\
I_3 &= \int_0^t \int [\partial_2 u^3 \partial_3 u^1 - \partial_1 u^3 \partial_3 u^2 - \partial_2 b^3 \partial_3 b^1 + \partial_1 b^3 \partial_3 b^2] \omega_{\frac{1}{2}} d\tau, \\
I_4 &= \int_0^t \int [-\partial_3 b^2 \partial_1 u^3 + \partial_3 b^1 \partial_2 u^3 + \partial_3 u^2 \partial_1 b^3 - \partial_3 u^1 \partial_2 b^3] d_{\frac{1}{2}} d\tau, \\
I_5 &= 2 \int_0^t \int (\partial_1 b^h \cdot \partial_2 u^h - \partial_2 b^h \cdot \partial_1 u^h) d_{\frac{1}{2}} d\tau.
\end{aligned}$$

Firstly, after integrating by parts we estimate

$$\begin{aligned}
I_1 &\lesssim \int_0^t \int |u^3| (|\omega|^{\frac{1}{2}} |\partial_3 \omega| + |d|^{\frac{1}{2}} |\partial_3 d|) d\tau \tag{28} \\
&\lesssim \int_0^t \|u^3\|_{L^{\frac{3p}{p-2}}} \left(\left(\int (|\omega|^{\frac{1}{2}})^{\frac{3p}{2}} \right)^{\frac{2}{3p}} \|\partial_3 \omega\|_{L^{\frac{3}{2}}} + \left(\int (|d|^{\frac{1}{2}})^{\frac{3p}{2}} \right)^{\frac{2}{3p}} \|\partial_3 d\|_{L^{\frac{3}{2}}} \right) d\tau \\
&\lesssim \int_0^t \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} (\|\omega_{\frac{3}{4}}\|_{L^p}^{\frac{2}{3}} \|\nabla \omega\|_{L^{\frac{3}{2}}} + \|d_{\frac{3}{4}}\|_{L^p}^{\frac{2}{3}} \|\nabla d\|_{L^{\frac{3}{2}}}) d\tau \\
&\lesssim \int_0^t \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} (\|\nabla \omega_{\frac{3}{4}}\|_{L^2}^{2(\frac{p-1}{p})} + \|\nabla d_{\frac{3}{4}}\|_{L^2}^{2(\frac{p-1}{p})}) (\|\omega_{\frac{3}{4}}\|_{L^2}^{\frac{2}{p}} + \|d_{\frac{3}{4}}\|_{L^2}^{\frac{2}{p}}) d\tau \\
&\leq \frac{1}{18} \int_0^t \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 + \|\nabla d_{\frac{3}{4}}\|_{L^2}^2 d\tau + c \int_0^t \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p (\|\omega_{\frac{3}{4}}\|_{L^2}^2 + \|d_{\frac{3}{4}}\|_{L^2}^2) d\tau
\end{aligned}$$

by Hölder's inequalities, Sobolev embedding of $\dot{H}^{\frac{1}{2}+\frac{2}{p}}(\mathbb{R}^3) \hookrightarrow L^{\frac{3p}{p-2}}(\mathbb{R}^3)$, Gagliardo-Nirenberg inequalities, (9) and Young's inequalities.

Next, we estimate

$$\begin{aligned}
I_2 &\lesssim \int_0^t \|\nabla b\|_{L^{p_2}} (\|d\|_{L^{\frac{3}{2}}} \left(\int |\omega|^{\frac{1}{2}(\frac{3p_2}{p_2-3})} \right)^{\frac{p_2-3}{3p_2}} + \|\omega\|_{L^{\frac{3}{2}}} \left(\int |d|^{\frac{1}{2}(\frac{3p_2}{p_2-3})} \right)^{\frac{p_2-3}{3p_2}}) d\tau \tag{29} \\
&\lesssim \int_0^t \|\nabla b\|_{L^{p_2}} (\|d_{\frac{3}{4}}\|_{L^2}^{\frac{4}{3}} \|\omega_{\frac{3}{4}}\|_{L^2}^{\frac{2p_2-9}{3p_2}} \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^{\frac{3}{p_2}} + \|\omega_{\frac{3}{4}}\|_{L^2}^{\frac{4}{3}} \|d_{\frac{3}{4}}\|_{L^2}^{\frac{2p_2-9}{3p_2}} \|\nabla d_{\frac{3}{4}}\|_{L^2}^{\frac{3}{p_2}}) d\tau \\
&\leq \frac{1}{18} \int_0^t \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 + \|\nabla d_{\frac{3}{4}}\|_{L^2}^2 d\tau + c \int_0^t \|\nabla b\|_{L^{p_2}}^{\frac{2p_2}{2p_2-3}} (\|d_{\frac{3}{4}}\|_{L^2}^2 + \|\omega_{\frac{3}{4}}\|_{L^2}^2) d\tau
\end{aligned}$$

by Hölder's, Gagliardo-Nirenberg and Young's inequalities.

Next, we first write by (5)

$$\begin{aligned}
I_3 &= \int_0^t \int [\partial_2 u^3 \partial_2 \Delta_h^{-1} (-\partial_3 \omega) + \partial_1 u^3 \partial_1 \Delta_h^{-1} (-\partial_3 \omega) \\
&\quad + \partial_2 b^3 \partial_2 \Delta_h^{-1} \partial_3 d + \partial_1 b^3 \partial_1 \Delta_h^{-1} \partial_3 d] \omega_{\frac{1}{2}} \\
&\quad + [\partial_2 u^3 \partial_1 \Delta_h^{-1} (-\partial_3^2 u^3) + \partial_1 u^3 \partial_2 \Delta_h^{-1} (\partial_3^2 u^3) \\
&\quad + \partial_2 b^3 \partial_1 \Delta_h^{-1} (\partial_3^2 b^3) - \partial_1 b^3 \partial_2 \Delta_h^{-1} (\partial_3^2 b^3)] \omega_{\frac{1}{2}} d\tau. \tag{30}
\end{aligned}$$

We bound the first and second terms of (30) as follows:

$$\begin{aligned}
& \int_0^t \int [\partial_2 u^3 \partial_2 \Delta_h^{-1} (-\partial_3 \omega) + \partial_1 u^3 \partial_1 \Delta_h^{-1} (-\partial_3 \omega)] \omega_{\frac{1}{2}} d\tau \\
& \lesssim \int_0^t \|\partial_3 \omega\|_{L^{\frac{3}{2}}} \|u^3\|_{\dot{H}^{\frac{3}{2}-\frac{2}{3}\sigma}} \|\omega_{\frac{3}{4}}\|_{\dot{H}^\sigma}^{\frac{2}{3}} d\tau \\
& \lesssim \int_0^t \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \|\omega_{\frac{3}{4}}\|_{L^2}^{\frac{2}{p}} \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^{2(1-\frac{1}{p})} d\tau \\
& \leq \frac{1}{36} \int_0^t \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 d\tau + c \int_0^t \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p \|\omega_{\frac{3}{4}}\|_{L^2}^2 d\tau
\end{aligned} \tag{31}$$

by (13a) with $f = \partial_3 \omega, g = u^3, h = \omega, \sigma = 3(\frac{1}{2} - \frac{1}{p})$, (9), Gagliardo-Nirenberg and Young's inequalities. Similarly we bound third and fourth terms of (30) by

$$\begin{aligned}
& \int_0^t \int [\partial_2 b^3 \partial_2 \Delta_h^{-1} (\partial_3 d) + \partial_1 b^3 \partial_1 \Delta_h^{-1} \partial_3 d] \omega_{\frac{1}{2}} d\tau \\
& \lesssim \int_0^t \|\partial_3 d\|_{L^{\frac{3}{2}}} \|b^3\|_{\dot{H}^{\frac{3}{2}-\frac{2}{3}\sigma}} \|\omega_{\frac{3}{4}}\|_{\dot{H}^\sigma}^{\frac{2}{3}} d\tau \\
& \lesssim \int_0^t \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \|\nabla d_{\frac{3}{4}}\|_{L^2} \|d_{\frac{3}{4}}\|_{L^2}^{\frac{1}{3}} \|\omega_{\frac{3}{4}}\|_{L^2}^{\frac{2}{p}-\frac{1}{3}} \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^{1-\frac{2}{p}} d\tau \\
& \leq \frac{1}{72} \int_0^t \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 + \|\nabla d_{\frac{3}{4}}\|_{L^2}^2 d\tau + c \int_0^t \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p (\|\omega_{\frac{3}{4}}\|_{L^2}^2 + \|d_{\frac{3}{4}}\|_{L^2}^2) d\tau
\end{aligned} \tag{32}$$

by (13a) with $f = \partial_3 d, g = b^3, h = \omega$, (9), $\sigma = 3(\frac{1}{2} - \frac{1}{p})$, Gagliardo-Nirenberg and Young's inequalities. Next, we bound the fifth, sixth, seventh and eighth terms of (30) by

$$\begin{aligned}
& \int_0^t \int [\partial_2 u^3 \partial_1 \Delta_h^{-1} (-\partial_3^2 u^3) + \partial_1 u^3 \partial_2 \Delta_h^{-1} (\partial_3^2 u^3) \\
& \quad + \partial_2 b^3 \partial_1 \Delta_h^{-1} (\partial_3^2 b^3) - \partial_1 b^3 \partial_2 \Delta_h^{-1} (\partial_3^2 b^3)] \omega_{\frac{1}{2}} d\tau \\
& \lesssim \int_0^t (\|\partial_3^2 u^3\|_{\mathcal{H}_\theta} + \|\partial_3^2 b^3\|_{\mathcal{H}_\theta}) (\|u^3\|_{\dot{H}^{\frac{3}{2}-\frac{2}{3}\sigma}} + \|b^3\|_{\dot{H}^{\frac{3}{2}-\frac{2}{3}\sigma}}) \|\omega_{\frac{3}{4}}\|_{\dot{H}^\sigma}^{\frac{2}{3}} d\tau \\
& \lesssim \int_0^t (\|\partial_3^2 u^3\|_{\mathcal{H}_\theta} + \|\partial_3^2 b^3\|_{\mathcal{H}_\theta}) (\|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}) \|\omega_{\frac{3}{4}}\|_{L^2}^{\frac{2}{3}(1-\sigma)} \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^{\frac{2\sigma}{3}} d\tau \\
& \lesssim \left(\int_0^t \|\partial_3^2 u^3\|_{\mathcal{H}_\theta}^2 + \|\partial_3^2 b^3\|_{\mathcal{H}_\theta}^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p d\tau \right)^{\frac{1}{6}} \\
& \quad \times \left(\int_0^t (\|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p) \|\omega_{\frac{3}{4}}\|_{L^2}^2 d\tau \right)^{\frac{1}{p}-\frac{1}{6}} \left(\int_0^t \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 d\tau \right)^{\frac{1}{2}-\frac{1}{p}} \\
& \leq \frac{1}{72} \int_0^t \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 d\tau + c \int_0^t (\|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p) \|\omega_{\frac{3}{4}}\|_{L^2}^2 d\tau \\
& \quad + c \left(\int_0^t \|\partial_3^2 u^3\|_{\mathcal{H}_\theta}^2 + \|\partial_3^2 b^3\|_{\mathcal{H}_\theta}^2 d\tau \right)^{\frac{3}{4}} \left(\int_0^t \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p d\tau \right)^{\frac{1}{4}}
\end{aligned} \tag{33}$$

by (13b) with $(f, g, h) = (\partial_3^2 u^3, u^3, \omega), (\partial_3^2 b^3, b^3, \omega), \sigma = 3(\frac{1}{2} - \frac{1}{p})$, Gagliardo-Nirenberg, Hölder's and Young's inequalities. Therefore, in sum of (31)-(33) in (30), we have

$$\begin{aligned}
I_3 &\leq \frac{1}{18} \int_0^t \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 + \|\nabla d_{\frac{3}{4}}\|_{L^2}^2 d\tau \\
&\quad + c \int_0^t (\|u^3\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}}^p + \|b^3\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}}^p) (\|\omega_{\frac{3}{4}}\|_{L^2}^2 + \|d_{\frac{3}{4}}\|_{L^2}^2) d\tau \\
&\quad + c \left(\int_0^t \|\partial_3^2 u^3\|_{\mathcal{H}_\theta}^2 + \|\partial_3^2 b^3\|_{\mathcal{H}_\theta}^2 d\tau \right)^{\frac{3}{4}} \left(\int_0^t \|u^3\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}}^p + \|b^3\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}}^p d\tau \right)^{\frac{1}{4}}.
\end{aligned} \tag{34}$$

Similarly, we can rewrite by (5) and then estimate

$$\begin{aligned}
I_4 &= \int_0^t \int [-\partial_1 u^3 \partial_1 \Delta_h^{-1} \partial_3 d - \partial_2 u^3 \partial_2 \Delta_h^{-1} \partial_3 d \\
&\quad + \partial_1 b^3 \partial_1 \Delta_h^{-1} \partial_3 \omega + \partial_2 b^3 \partial_2 \Delta_h^{-1} \partial_3 \omega] d_{\frac{1}{2}} d\tau \\
&\quad + [\partial_1 u^3 \partial_2 \Delta_h^{-1} \partial_3^2 b^3 - \partial_2 u^3 \partial_1 \Delta_h^{-1} \partial_3^2 b^3 \\
&\quad - \partial_1 b^3 \partial_2 \Delta_h^{-1} \partial_3^2 u^3 + \partial_2 b^3 \partial_1 \Delta_h^{-1} \partial_3^2 u^3] d_{\frac{1}{2}} d\tau \\
&\lesssim \int_0^t (\|\partial_3 d\|_{L^{\frac{3}{2}}} + \|\partial_3 \omega\|_{L^{\frac{3}{2}}}) (\|u^3\|_{\dot{H}^{\frac{3}{2}-\frac{2}{3}\sigma}} + \|b^3\|_{\dot{H}^{\frac{3}{2}-\frac{2}{3}\sigma}}) \|d_{\frac{3}{4}}\|_{\dot{H}^\sigma}^{\frac{2}{3}} \\
&\quad + (\|\partial_3^2 b^3\|_{\mathcal{H}_\theta} + \|\partial_3^2 u^3\|_{\mathcal{H}_\theta}) (\|u^3\|_{\dot{H}^{\frac{3}{2}-\frac{2}{3}\sigma}} + \|b^3\|_{\dot{H}^{\frac{3}{2}-\frac{2}{3}\sigma}}) \|d_{\frac{3}{4}}\|_{\dot{H}^\sigma}^{\frac{2}{3}} d\tau \\
&\lesssim \int_0^t \left(\|\nabla d_{\frac{3}{4}}\|_{L^2}^2 + \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 \right)^{1-\frac{1}{p}} \left(\|\omega_{\frac{3}{4}}\|_{L^2} + \|d_{\frac{3}{4}}\|_{L^2} \right)^{\frac{1}{3}} \\
&\quad \times \left(\|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \right) \|d_{\frac{3}{4}}\|_{L^2}^{-\frac{1}{3}+\frac{2}{p}} \\
&\quad + (\|\partial_3^2 u^3\|_{\mathcal{H}_\theta} + \|\partial_3^2 b^3\|_{\mathcal{H}_\theta}) (\|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}) \|d_{\frac{3}{4}}\|_{L^2}^{-\frac{1}{3}+\frac{2}{p}} \|\nabla d_{\frac{3}{4}}\|_{L^2}^{1-\frac{2}{p}} d\tau \\
&\lesssim \int_0^t \left(\|\nabla d_{\frac{3}{4}}\|_{L^2}^2 + \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 \right)^{1-\frac{1}{p}} \left(\|\omega_{\frac{3}{4}}\|_{L^2}^2 + \|d_{\frac{3}{4}}\|_{L^2}^2 \right)^{\frac{1}{p}} \\
&\quad \times \left(\|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \right) \\
&\quad + (\|\partial_3^2 u^3\|_{\mathcal{H}_\theta} + \|\partial_3^2 b^3\|_{\mathcal{H}_\theta}) \left(\|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p \right)^{\frac{1}{6}} \\
&\quad \times [(\|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p) (\|\omega_{\frac{3}{4}}\|_{L^2}^2 + \|d_{\frac{3}{4}}\|_{L^2}^2)]^{\frac{1}{p}-\frac{1}{6}} \|\nabla d_{\frac{3}{4}}\|_{L^2}^{2(\frac{1}{2}-\frac{1}{p})} d\tau \\
&\leq \frac{1}{36} \int_0^t \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 + \|\nabla d_{\frac{3}{4}}\|_{L^2}^2 d\tau \\
&\quad + c \int_0^t (\|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p) (\|\omega_{\frac{3}{4}}\|_{L^2}^2 + \|d_{\frac{3}{4}}\|_{L^2}^2) d\tau \\
&\quad + c \left(\int_0^t \|\partial_3^2 u^3\|_{\mathcal{H}_\theta}^2 + \|\partial_3^2 b^3\|_{\mathcal{H}_\theta}^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p d\tau \right)^{\frac{1}{6}} \\
&\quad \times \left(\int_0^t (\|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}) (\|\omega_{\frac{3}{4}}\|_{L^2}^2 + \|d_{\frac{3}{4}}\|_{L^2}^2) d\tau \right)^{\frac{1}{p}-\frac{1}{6}} \left(\int_0^t \|\nabla d_{\frac{3}{4}}\|_{L^2}^2 d\tau \right)^{\frac{1}{2}-\frac{1}{p}} \\
&\leq \frac{1}{18} \int_0^t \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 + \|\nabla d_{\frac{3}{4}}\|_{L^2}^2 d\tau \\
&\quad + c \int_0^t (\|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p) (\|\omega_{\frac{3}{4}}\|_{L^2}^2 + \|d_{\frac{3}{4}}\|_{L^2}^2) d\tau \\
&\quad + c \left(\int_0^t \|\partial_3^2 u^3\|_{\mathcal{H}_\theta}^2 + \|\partial_3^2 b^3\|_{\mathcal{H}_\theta}^2 d\tau \right)^{\frac{3}{4}} \left(\int_0^t \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p d\tau \right)^{\frac{1}{4}}
\end{aligned} \tag{35}$$

by (13a) with $(f, g, h) = (\partial_3 d, u^3, d)$, $(\partial_3 \omega, b^3, d)$, (13b) with $(f, g, h) = (\partial_3^2 b^3, u^3, d)$, $(\partial_3^2 u^3, b^3, d)$, $\sigma = 3(\frac{1}{2} - \frac{1}{p})$, (9), Gagliardo-Nirenberg, Young's and Hölder's inequalities.

We now work on the term with all index being one's and two's. We write by (5)

$$I_5 = 2 \int_0^t \int [\partial_1 b^h \cdot \partial_2 (\nabla_h^\perp \Delta_h^{-1} \omega - \nabla_h \Delta_h^{-1} \partial_3 u^3) - \partial_2 b^h \cdot \partial_1 (\nabla_h^\perp \Delta_h^{-1} \omega - \nabla_h \Delta_h^{-1} \partial_3 u^3)] d_{\frac{1}{2}} d\tau. \quad (36)$$

We bound by identical estimates applied in (29) to obtain

$$\begin{aligned} & 2 \int_0^t \int [\partial_1 b^h \cdot \partial_2 \nabla_h^\perp \Delta_h^{-1} \omega - \partial_2 b^h \cdot \partial_1 \nabla_h^\perp \Delta_h^{-1} \omega] d_{\frac{1}{2}} d\tau \\ & \lesssim \int_0^t \|\nabla b\|_{L^{p_2}} \|\omega\|_{L^{\frac{3}{2}}} \|d_{\frac{3}{4}}\|_{L^{\frac{2p_2}{p_2-3}}}^{\frac{2}{3}} d\tau \\ & \lesssim \int_0^t \|\nabla b\|_{L^{p_2}} \|\omega_{\frac{3}{4}}\|_{L^2}^{\frac{4}{3}} \|d_{\frac{3}{4}}\|_{L^2}^{\frac{2p_2-9}{3p_2}} \|\nabla d_{\frac{3}{4}}\|_{L^2}^{\frac{3}{p_2}} d\tau \\ & \leq \frac{1}{36} \int_0^t \|\nabla d_{\frac{3}{4}}\|_{L^2}^2 + c \int_0^t \|\nabla b\|_{L^{p_2}}^{\frac{2p_2}{2p_2-3}} (\|\omega_{\frac{3}{4}}\|_{L^2}^2 + \|d_{\frac{3}{4}}\|_{L^2}^2) d\tau \end{aligned} \quad (37)$$

by Hölder's, Gagliardo-Nirenberg and Young's inequalities while we bound

$$\begin{aligned} & 2 \int_0^t \int [-\partial_1 b^h \cdot \partial_2 \nabla_h \Delta_h^{-1} \partial_3 u^3 + \partial_2 b^h \cdot \partial_1 \nabla_h \Delta_h^{-1} \partial_3 u^3] d_{\frac{1}{2}} d\tau \\ & \lesssim \int_0^t \|\nabla_h \partial_3 u^3\|_{\mathcal{H}_\theta} \|b^h\|_{\dot{H}^{\frac{3}{2}-\frac{2}{p}}} \|d_{\frac{3}{4}}\|_{\dot{H}^\sigma}^{\frac{2}{3}} d\tau \\ & \lesssim \int_0^t \|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta} \|b\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^{\frac{p}{6}} (\|b\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p \|d_{\frac{3}{4}}\|_{L^2}^2)^{\frac{1}{p}-\frac{1}{6}} \|\nabla d_{\frac{3}{4}}\|_{L^2}^{2(\frac{1}{2}-\frac{1}{p})} d\tau \\ & \lesssim \left(\int_0^t \|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta}^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t \|b\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p d\tau \right)^{\frac{1}{6}} \\ & \quad \times \left(\int_0^t \|b\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p \|d_{\frac{3}{4}}\|_{L^2}^2 d\tau \right)^{\frac{1}{p}-\frac{1}{6}} \left(\int_0^t \|\nabla d_{\frac{3}{4}}\|_{L^2}^2 d\tau \right)^{\frac{1}{2}-\frac{1}{p}} \\ & \leq \frac{1}{36} \int_0^t \|\nabla d_{\frac{3}{4}}\|_{L^2}^2 d\tau \\ & \quad + c \left(\int_0^t \|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta}^2 d\tau \right)^{\frac{3}{4}} \left(\int_0^t \|b\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p d\tau \right)^{\frac{1}{4}} + c \int_0^t \|b\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p \|d_{\frac{3}{4}}\|_{L^2}^2 d\tau \end{aligned} \quad (38)$$

by (13b) with $(f, g, h) = (\nabla_h \partial_3 u^3, b^h, d)$, $\sigma = 3(\frac{1}{2} - \frac{1}{p})$, Gagliardo-Nirenberg, Hölder's and Young's inequalities. Thus, due to (37), (38) in (36)

$$\begin{aligned} I_5 & \leq \frac{1}{18} \int_0^t \|\nabla d_{\frac{3}{4}}\|_{L^2}^2 d\tau + c \int_0^t (\|b\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|\nabla b\|_{L^{p_2}}^{\frac{2p_2}{2p_2-3}}) (\|\omega_{\frac{3}{4}}\|_{L^2}^2 + \|d_{\frac{3}{4}}\|_{L^2}^2) d\tau \\ & \quad + c \left(\int_0^t \|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta}^2 d\tau \right)^{\frac{3}{4}} \left(\int_0^t \|b\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p d\tau \right)^{\frac{1}{4}}. \end{aligned} \quad (39)$$

In sum of (26)-(29), (34), (35), (39), we have

$$\begin{aligned}
& \frac{2}{3}(\|\omega_{\frac{3}{4}}\|_{L^2}^2 + \|d_{\frac{3}{4}}\|_{L^2}^2)(t) + \frac{5}{9} \int_0^t \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 + \|\nabla d_{\frac{3}{4}}\|_{L^2}^2 d\tau \\
& \leq \frac{2}{3}(\|\omega_{\frac{3}{4}}\|_{L^2}^2 + \|d_{\frac{3}{4}}\|_{L^2}^2)(0) \\
& \quad + c \int_0^t (\|\omega_{\frac{3}{4}}\|_{L^2}^2 + \|d_{\frac{3}{4}}\|_{L^2}^2)(\|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b\|_{L^{\frac{2p_1}{p_1-3}}}^{\frac{2p_1}{p_1-3}} + \|\nabla b\|_{L^{\frac{2p_2}{p_2-3}}}^{\frac{2p_2}{p_2-3}}) d\tau \\
& \quad + c \left(\int_0^t \|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta}^2 + \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta}^2 d\tau \right)^{\frac{3}{4}} \left(\int_0^t \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p d\tau \right)^{\frac{1}{4}}
\end{aligned} \tag{40}$$

so that Gronwall's type inequality argument using

$$\begin{aligned}
& e^{c \int_0^t \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b\|_{L^{\frac{2p_1}{p_1-3}}}^{\frac{2p_1}{p_1-3}} + \|\nabla b\|_{L^{\frac{2p_2}{p_2-3}}}^{\frac{2p_2}{p_2-3}} d\tau} \left(\int_0^t \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p d\tau \right)^{\frac{1}{4}} \\
& \lesssim e^{c \int_0^t \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b\|_{L^{\frac{2p_1}{p_1-3}}}^{\frac{2p_1}{p_1-3}} + \|\nabla b\|_{L^{\frac{2p_2}{p_2-3}}}^{\frac{2p_2}{p_2-3}} d\tau}
\end{aligned}$$

completes the proof of Proposition 3.1. \square

Proposition 3.2. *Under the hypothesis of Theorem 1.1, for $\theta \in (\frac{1}{2} - \frac{2}{p}, \frac{1}{6})$, the solution to the MHD system (1a)-(1c) satisfies for any $t < T^*$*

$$\begin{aligned}
& (\|\partial_3 u^3\|_{\mathcal{H}_\theta}^2 + \|\partial_3 b^3\|_{\mathcal{H}_\theta}^2)(t) + \int_0^t \|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta}^2 + \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta}^2 d\tau \\
& \lesssim e^{c \int_0^t \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p d\tau} \\
& \times [\|\Omega_0\|_{L^{\frac{3}{2}}}^2 + \|j_0\|_{L^{\frac{3}{2}}}^2 \\
& \quad + \int_0^t \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^2 (\|\omega_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+3}{3p})} \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^{2(1-\frac{1}{p})} + \|d_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+3}{3p})} \|\nabla d_{\frac{3}{4}}\|_{L^2}^{2(1-\frac{1}{p})}) \\
& \quad + (\|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^2 + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^2)(\|\omega_{\frac{3}{4}}\|_{L^2}^2 + \|d_{\frac{3}{4}}\|_{L^2}^2)^{\frac{p+6}{3p}} (\|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 + \|\nabla d_{\frac{3}{4}}\|_{L^2}^2)^{1-\frac{2}{p}} d\tau].
\end{aligned}$$

Proof. Applying ∂_3 on the third components of (1a), (1b), we obtain

$$\begin{aligned}
\partial_t \partial_3 u^3 - \Delta \partial_3 u^3 &= -\partial_3 u \cdot \nabla u^3 - (u \cdot \nabla) \partial_3 u^3 + \partial_3 b \cdot \nabla b^3 + (b \cdot \nabla) \partial_3 b^3 \\
&\quad - \partial_3^2 (-\Delta)^{-1} \sum_{l,m=1}^3 (\partial_l u^m \partial_m u^l - \partial_l b^m \partial_m b^l),
\end{aligned} \tag{41a}$$

$$\partial_t \partial_3 b^3 - \Delta \partial_3 b^3 = -\partial_3 u \cdot \nabla b^3 - (u \cdot \nabla) \partial_3 b^3 + \partial_3 b \cdot \nabla u^3 + (b \cdot \nabla) \partial_3 u^3. \tag{41b}$$

We write

$$\begin{aligned}
& \sum_{l,m=1}^3 \partial_l u^m \partial_m u^l - \partial_l b^m \partial_m b^l \\
& = \sum_{l,m=1}^2 \partial_l u^m \partial_m u^l - \partial_l b^m \partial_m b^l + 2 \sum_{l=1}^2 \partial_l u^3 \partial_3 u^l - \partial_l b^3 \partial_3 b^l + (\partial_3 u^3)^2 - (\partial_3 b^3)^2,
\end{aligned}$$

and

$$\begin{aligned} -\partial_3 u \cdot \nabla u^3 + \partial_3 b \cdot \nabla b^3 &= \sum_{l=1}^2 -\partial_3 u^l \partial_l u^3 + \partial_3 b^l \partial_l b^3 - (\partial_3 u^3)^2 + (\partial_3 b^3)^2, \\ -\partial_3 u \cdot \nabla b^3 + \partial_3 b \cdot \nabla u^3 &= \sum_{l=1}^2 -\partial_3 u^l \partial_l b^3 + \partial_3 b^l \partial_l u^3 \end{aligned}$$

so that we can take \mathcal{H}_θ -inner products on (41a), (41b) and sum to obtain

$$\begin{aligned} &\frac{1}{2} \partial_t (\|\partial_3 u^3\|_{\mathcal{H}_\theta}^2 + \|\partial_3 b^3\|_{\mathcal{H}_\theta}^2) + \|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta}^2 + \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta}^2 \\ &= \sum_{n=1}^3 (Q_n(u, b) | \partial_3 u^3)_{\mathcal{H}_\theta} + \sum_{n=1}^2 (R_n(u, b) | \partial_3 b^3)_{\mathcal{H}_\theta} \end{aligned} \quad (42)$$

where

$$\begin{aligned} Q_1(u, b) &\triangleq (-Id - \partial_3^2 (-\Delta)^{-1}) ((\partial_3 u^3)^2 - (\partial_3 b^3)^2) \\ &\quad - \partial_3^2 (-\Delta)^{-1} \sum_{l,m=1}^2 (\partial_l u^m \partial_m u^l - \partial_l b^m \partial_m b^l), \end{aligned} \quad (43a)$$

$$Q_2(u, b) \triangleq (Id + 2\partial_3^2 (-\Delta)^{-1}) \left(\sum_{l=1}^2 -\partial_3 u^l \partial_l u^3 + \partial_3 b^l \partial_l b^3 \right), \quad (43b)$$

$$Q_3(u, b) \triangleq -(u \cdot \nabla) \partial_3 u^3 + (b \cdot \nabla) \partial_3 b^3, \quad (43c)$$

$$R_1(u, b) \triangleq \sum_{l=1}^2 -\partial_3 u^l \partial_l b^3 + \partial_3 b^l \partial_l u^3, \quad (43d)$$

$$R_2(u, b) \triangleq -(u \cdot \nabla) \partial_3 b^3 + (b \cdot \nabla) \partial_3 u^3. \quad (43e)$$

We estimate first

$$\begin{aligned} &|(Q_1(u, b) | \partial_3 u^3)_{\mathcal{H}_\theta}| \\ &\leq |(-Id - \partial_3^2 (-\Delta)^{-1}) ((\partial_3 u^3)^2 - (\partial_3 b^3)^2) | \partial_3 u^3)_{\mathcal{H}_\theta}| \\ &\quad + |(\partial_3^2 (-\Delta)^{-1} \sum_{l,m=1}^2 (\partial_l u^m \partial_m u^l - \partial_l b^m \partial_m b^l) | \partial_3 u^3)_{\mathcal{H}_\theta}| \triangleq II_1 + II_2 \end{aligned} \quad (44)$$

where

$$\begin{aligned} II_1 &\lesssim \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^2 (\|\partial_3 u^3\|_{\dot{H}^{\theta, \frac{1}{2}-\theta-\frac{1}{p}}}^2 + \|\partial_3 b^3\|_{\dot{H}^{\theta, \frac{1}{2}-\theta-\frac{1}{p}}}^2) \\ &\lesssim \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^2 (\|\partial_3 u^3\|_{\mathcal{H}_\theta}^{\frac{2}{p}} \|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta}^{2(1-\frac{1}{p})} + \|\partial_3 b^3\|_{\mathcal{H}_\theta}^{\frac{2}{p}} \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta}^{2(1-\frac{1}{p})}) \\ &\leq \frac{1}{36} (\|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta}^2 + \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta}^2) + c \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p (\|\partial_3 u^3\|_{\mathcal{H}_\theta}^2 + \|\partial_3 b^3\|_{\mathcal{H}_\theta}^2) \end{aligned} \quad (45)$$

by (14) with $A = -Id - \partial_3^2(-\Delta)^{-1}$, $f = g = \partial_3 u^3$, $h = u^3$ and again with $f = g = \partial_3 b^3$, $h = u^3$ for θ such that $0 < \theta < \frac{1}{2} - \frac{1}{p}$, (12) and Young's inequality. Similarly,

$$\begin{aligned}
II_2 &\lesssim \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \\
&\quad \times (\|\omega\|_{\dot{H}^{\theta, \frac{1}{2}-\theta-\frac{1}{p}}}^2 + \|\partial_3 u^3\|_{\dot{H}^{\theta, \frac{1}{2}-\theta-\frac{1}{p}}}^2 + \|d\|_{\dot{H}^{\theta, \frac{1}{2}-\theta-\frac{1}{p}}}^2 + \|\partial_3 b^3\|_{\dot{H}^{\theta, \frac{1}{2}-\theta-\frac{1}{p}}}^2) \\
&\lesssim \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} (\|\omega_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+3}{3p})} \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^{2(1-\frac{1}{p})} + \|d_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+3}{3p})} \|\nabla d_{\frac{3}{4}}\|_{L^2}^{2(1-\frac{1}{p})} \\
&\quad + \|\partial_3 u^3\|_{\mathcal{H}_\theta}^{\frac{2}{p}} \|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta}^{2-\frac{2}{p}} + \|\partial_3 b^3\|_{\mathcal{H}_\theta}^{\frac{2}{p}} \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta}^{2-\frac{2}{p}}) \\
&\leq \frac{1}{36} (\|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta}^2 + \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta}^2) + c \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p (\|\partial_3 u^3\|_{\mathcal{H}_\theta}^2 + \|\partial_3 b^3\|_{\mathcal{H}_\theta}^2) \\
&\quad + c \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} (\|\omega_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+3}{3p})} \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^{2(1-\frac{1}{p})} + \|d_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+3}{3p})} \|\nabla d_{\frac{3}{4}}\|_{L^2}^{2(1-\frac{1}{p})})
\end{aligned} \tag{46}$$

by (5), (14), (15), (12) and Young's inequality.

We now consider

$$\begin{aligned}
(Q_2(u, b)|\partial_3 u^3)_{\mathcal{H}_\theta} &= ((Id + 2\partial_3^2(-\Delta)^{-1})(\sum_{l=1}^2 u^l \partial_l u^3 - b^l \partial_l b^3)|\partial_3^2 u^3)_{\mathcal{H}_\theta} \\
&\quad + ((Id + 2\partial_3^2(-\Delta)^{-1})(\sum_{l=1}^2 u^l \partial_l \partial_3 u^3 - b^l \partial_l \partial_3 b^3)|\partial_3 u^3)_{\mathcal{H}_\theta} \\
&\triangleq (Q_{2,1}(u, b)|\partial_3 u^3)_{\mathcal{H}_\theta} + (Q_{2,2}(u, b)|\partial_3 u^3)_{\mathcal{H}_\theta}
\end{aligned} \tag{47}$$

due to integration by parts. We estimate

$$\begin{aligned}
&|(Q_{2,1}(u, b)|\partial_3 u^3)_{\mathcal{H}_\theta}| \\
&\lesssim \sum_{l=1}^2 (\|u^l \partial_l u^3\|_{\dot{H}^{-\frac{1}{2}+\theta, -\theta}} + \|b^l \partial_l b^3\|_{\dot{H}^{-\frac{1}{2}+\theta, -\theta}}) \|\partial_3^2 u^3\|_{\mathcal{H}_\theta} \\
&\lesssim \sum_{l=1}^2 (\|u^l\|_{(\dot{B}_{2,1}^1)_h(\dot{B}_{2,1}^{\frac{1}{2}-\frac{2}{p}})_v} \|\partial_l u^3\|_{\dot{H}^{-\frac{1}{2}+\theta, \frac{2}{p}-\theta}} \\
&\quad + \|b^l\|_{(\dot{B}_{2,1}^1)_h(\dot{B}_{2,1}^{\frac{1}{2}-\frac{2}{p}})_v} \|\partial_l b^3\|_{\dot{H}^{-\frac{1}{2}+\theta, \frac{2}{p}-\theta}}) \|\partial_3^2 u^3\|_{\mathcal{H}_\theta} \\
&\lesssim \sum_{l=1}^2 (\|u^l\|_{(\dot{B}_{2,1}^1)_h(\dot{B}_{2,1}^{\frac{1}{2}-\frac{2}{p}})_v} \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} + \|b^l\|_{(\dot{B}_{2,1}^1)_h(\dot{B}_{2,1}^{\frac{1}{2}-\frac{2}{p}})_v} \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}) \|\partial_3^2 u^3\|_{\mathcal{H}_\theta} \\
&\lesssim \left(\|\omega_{\frac{3}{4}}\|_{L^2}^{\frac{1}{3}+\frac{2}{p}} \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^{1-\frac{2}{p}} + \|\partial_3 u^3\|_{\mathcal{H}_\theta}^{\frac{2}{p}} \|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta}^{1-\frac{2}{p}} \right) \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \|\partial_3^2 u^3\|_{\mathcal{H}_\theta} \\
&\quad + \left(\|d_{\frac{3}{4}}\|_{L^2}^{\frac{1}{3}+\frac{2}{p}} \|\nabla d_{\frac{3}{4}}\|_{L^2}^{1-\frac{2}{p}} + \|\partial_3 b^3\|_{\mathcal{H}_\theta}^{\frac{2}{p}} \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta}^{1-\frac{2}{p}} \right) \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \|\partial_3^2 u^3\|_{\mathcal{H}_\theta} \\
&\leq \frac{1}{36} (\|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta}^2 + \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta}^2) \\
&\quad + c \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^2 \|\omega_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+6}{3p})} \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^{2(1-\frac{2}{p})} + \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p \|\partial_3 u^3\|_{\mathcal{H}_\theta}^2 \\
&\quad + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^2 \|d_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+6}{3p})} \|\nabla d_{\frac{3}{4}}\|_{L^2}^{2(1-\frac{2}{p})} + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p \|\partial_3 b^3\|_{\mathcal{H}_\theta}^2
\end{aligned} \tag{48}$$

where we used continuity of Riesz transform, (94), the fact that $\sum_{l=1}^2 \|\partial_l f\|_{\dot{H}^{-\frac{1}{2}+\theta, \frac{2}{p}-\theta}} \lesssim \|f\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}$, Lemma 2.2 with $\alpha = \frac{2}{p}$ and Young's inequalities. Next,

$$\begin{aligned}
& |(Q_{2,2}(u, b)|\partial_3 u^3)_{\mathcal{H}_\theta}| \\
& \lesssim (\|\omega_{\frac{3}{4}}\|_{L^2}^{\frac{1}{3}+\frac{2}{p}} \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^{1-\frac{2}{p}} + \|\partial_3 u^3\|_{\mathcal{H}_\theta}^{\frac{2}{p}} \|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta}^{1-\frac{2}{p}}) \|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta} \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \\
& \quad + (\|d_{\frac{3}{4}}\|_{L^2}^{\frac{1}{3}+\frac{2}{p}} \|\nabla d_{\frac{3}{4}}\|_{L^2}^{1-\frac{2}{p}} + \|\partial_3 b^3\|_{\mathcal{H}_\theta}^{\frac{2}{p}} \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta}^{1-\frac{2}{p}}) \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta} \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \\
& \leq \frac{1}{36} (\|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta}^2 + \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta}^2) \\
& \quad + c(\|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^2 \|\omega_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+6}{3p})} \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^{2(1-\frac{2}{p})} + \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p \|\partial_3 u^3\|_{\mathcal{H}_\theta}^2) \\
& \quad + \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^2 \|d_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+6}{3p})} \|\nabla d_{\frac{3}{4}}\|_{L^2}^{2(1-\frac{2}{p})} + \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p \|\partial_3 b^3\|_{\mathcal{H}_\theta}^2)
\end{aligned} \tag{49}$$

by (16) with $A = Id + 2\partial_3^2(-\Delta)^{-1}$, $g = u, h = u$, (17) with $A = Id + 2\partial_3^2(-\Delta)^{-1}$, $g = b, h = u$ and Young's inequalities.

We treat R_1 similarly:

$$\begin{aligned}
& (R_1(u, b)|\partial_3 b^3)_{\mathcal{H}_\theta} \\
& = (\sum_{l=1}^2 u^l \partial_l b^3 - b^l \partial_l u^3 |\partial_3^2 b^3)_{\mathcal{H}_\theta} + (\sum_{l=1}^2 u^l \partial_l \partial_3 b^3 - b^l \partial_l \partial_3 u^3 |\partial_3 b^3)_{\mathcal{H}_\theta} \\
& \triangleq (R_{1,1}(u, b)|\partial_3^2 b^3)_{\mathcal{H}_\theta} + (R_{1,2}(u, b)|\partial_3 b^3)_{\mathcal{H}_\theta}
\end{aligned} \tag{50}$$

where

$$\begin{aligned}
& |(R_{1,1}(u, b)|\partial_3^2 b^3)_{\mathcal{H}_\theta}| \\
& \lesssim \sum_{l=1}^2 (\|u^l \partial_l b^3\|_{\mathcal{H}_\theta} + \|b^l \partial_l u^3\|_{\mathcal{H}_\theta}) \|\partial_3^2 u^3\|_{\mathcal{H}_\theta} \\
& \lesssim \sum_{l=1}^2 (\|u^l\|_{(\dot{B}_{2,1}^1)_h(\dot{B}_{2,1}^{\frac{1}{2}-\frac{2}{p}})_v} \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} + \|b^l\|_{(\dot{B}_{2,1}^1)_h(\dot{B}_{2,1}^{\frac{1}{2}-\frac{2}{p}})_v} \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}) \|\partial_3^2 u^3\|_{\mathcal{H}_\theta} \\
& \lesssim (\|\omega_{\frac{3}{4}}\|_{L^2}^{\frac{1}{3}+\frac{2}{p}} \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^{1-\frac{2}{p}} + \|\partial_3 u^3\|_{\mathcal{H}_\theta}^{\frac{2}{p}} \|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta}^{1-\frac{2}{p}}) \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \|\partial_3^2 u^3\|_{\mathcal{H}_\theta} \\
& \quad + (\|d_{\frac{3}{4}}\|_{L^2}^{\frac{1}{3}+\frac{2}{p}} \|\nabla d_{\frac{3}{4}}\|_{L^2}^{1-\frac{2}{p}} + \|\partial_3 b^3\|_{\mathcal{H}_\theta}^{\frac{2}{p}} \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta}^{1-\frac{2}{p}}) \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \|\partial_3^2 u^3\|_{\mathcal{H}_\theta} \\
& \leq \frac{1}{36} (\|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta}^2 + \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta}^2) \\
& \quad + c(\|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^2 \|\omega_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+6}{3p})} \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^{2(1-\frac{2}{p})} + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p \|\partial_3 u^3\|_{\mathcal{H}_\theta}^2) \\
& \quad + \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^2 \|d_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+6}{3p})} \|\nabla d_{\frac{3}{4}}\|_{L^2}^{2(1-\frac{2}{p})} + \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p \|\partial_3 b^3\|_{\mathcal{H}_\theta}^2)
\end{aligned} \tag{51}$$

by (94), Lemma 2.2 with $\alpha = \frac{2}{p}$ and Young's inequalities. Next, we work on

$$\begin{aligned}
& |(R_{1,2}(u, b)|\partial_3 b^3)_{\mathcal{H}_\theta}| \\
& \lesssim \left(\|\omega_{\frac{3}{4}}\|_{L^2}^{\frac{1}{3}+\frac{2}{p}} \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^{1-\frac{2}{p}} + \|\partial_3 u^3\|_{\mathcal{H}_\theta}^{\frac{2}{p}} \|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta}^{1-\frac{2}{p}} \right) \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta} \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \\
& \quad + \left(\|d_{\frac{3}{4}}\|_{L^2}^{\frac{1}{3}+\frac{2}{p}} \|\nabla d_{\frac{3}{4}}\|_{L^2}^{1-\frac{2}{p}} + \|\partial_3 b^3\|_{\mathcal{H}_\theta}^{\frac{2}{p}} \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta}^{1-\frac{2}{p}} \right) \|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta} \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \\
& \leq \frac{1}{36} (\|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta}^2 + \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta}^2) \\
& \quad + c(\|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^2 \|\omega_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+6}{3p})} \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^{2(1-\frac{2}{p})} + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p \|\partial_3 u^3\|_{\mathcal{H}_\theta}^2) \\
& \quad + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^2 \|d_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+6}{3p})} \|\nabla d_{\frac{3}{4}}\|_{L^2}^{2(1-\frac{2}{p})} + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p \|\partial_3 b^3\|_{\mathcal{H}_\theta}^2)
\end{aligned} \tag{52}$$

by (16) with $A = Id, g = b, h = b$ and (17) with $A = Id, g = u, h = b$ and Young's inequalities.

We finally work on

$$\begin{aligned}
& (Q_3(u, b)|\partial_3 u^3)_{\mathcal{H}_\theta} + (R_2(u, b)|\partial_3 b^3)_{\mathcal{H}_\theta} \\
& = (-u^h \cdot \nabla_h \partial_3 u^3 + b^h \cdot \nabla_h \partial_3 b^3 |\partial_3 u^3)_{\mathcal{H}_\theta} + (-u^3 \partial_3^2 u^3 + b^3 \partial_3^2 b^3 |\partial_3 u^3)_{\mathcal{H}_\theta} \\
& \quad + (-u^h \cdot \nabla_h \partial_3 b^3 + b^h \cdot \nabla_h \partial_3 u^3 |\partial_3 b^3)_{\mathcal{H}_\theta} + (-u^3 \partial_3^2 b^3 + b^3 \partial_3^2 u^3 |\partial_3 b^3)_{\mathcal{H}_\theta} \\
& \triangleq (Q_{3,1}(u, b)|\partial_3 u^3)_{\mathcal{H}_\theta} + (Q_{3,2}(u, b)|\partial_3 u^3)_{\mathcal{H}_\theta} + (R_{2,1}(u, b)|\partial_3 b^3)_{\mathcal{H}_\theta} + (R_{2,2}(u, b)|\partial_3 b^3)_{\mathcal{H}_\theta}
\end{aligned} \tag{53}$$

where

$$\begin{aligned}
& (Q_{3,1}(u, b)|\partial_3 u^3)_{\mathcal{H}_\theta} + (R_{2,1}(u, b)|\partial_3 b^3)_{\mathcal{H}_\theta} \\
& \lesssim (\|\omega_{\frac{3}{4}}\|_{L^2}^{\frac{1}{3}+\frac{2}{p}} \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^{1-\frac{2}{p}} + \|\partial_3 u^3\|_{\mathcal{H}_\theta}^{\frac{2}{p}} \|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta}^{1-\frac{2}{p}} \\
& \quad + \|d_{\frac{3}{4}}\|_{L^2}^{\frac{1}{3}+\frac{2}{p}} \|\nabla d_{\frac{3}{4}}\|_{L^2}^{1-\frac{2}{p}} + \|\partial_3 b^3\|_{\mathcal{H}_\theta}^{\frac{2}{p}} \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta}^{1-\frac{2}{p}}) \\
& \quad \times (\|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta} \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} + \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta} \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}) \\
& \quad + \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta} \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} + \|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta} \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}})
\end{aligned} \tag{54}$$

by (16) with $A = Id, (g, h) = (u, u), (b, b)$ and (17) with $A = Id, (g, h) = (b, u), (u, b)$. On the other hand,

$$\begin{aligned}
& (Q_{3,2}(u, b)|\partial_3 u^3)_{\mathcal{H}_\theta} + (R_{2,2}(u, b)|\partial_3 b^3)_{\mathcal{H}_\theta} \\
& \lesssim (\|u^3\|_{(\dot{B}_{2,2}^{\frac{2}{p}})_h(\dot{B}_{2,1}^{\frac{1}{2}})_v} \|\partial_3^2 u^3\|_{\dot{H}^{-\frac{1}{2}+\theta, -\theta}} \\
& \quad + \|b^3\|_{(\dot{B}_{2,2}^{\frac{2}{p}})_h(\dot{B}_{2,1}^{\frac{1}{2}})_v} \|\partial_3^2 b^3\|_{\dot{H}^{-\frac{1}{2}+\theta, -\theta}}) \|\partial_3 u^3\|_{\dot{H}^{\frac{1}{2}+\theta-\frac{2}{p}, -\theta}} \\
& \quad + (\|u^3\|_{(\dot{B}_{2,2}^{\frac{2}{p}})_h(\dot{B}_{2,1}^{\frac{1}{2}})_v} \|\partial_3^2 b^3\|_{\dot{H}^{-\frac{1}{2}+\theta, -\theta}} \\
& \quad + \|b^3\|_{(\dot{B}_{2,2}^{\frac{2}{p}})_h(\dot{B}_{2,1}^{\frac{1}{2}})_v} \|\partial_3^2 u^3\|_{\dot{H}^{-\frac{1}{2}+\theta, -\theta}}) \|\partial_3 b^3\|_{\dot{H}^{\frac{1}{2}+\theta-\frac{2}{p}, -\theta}} \\
& \lesssim \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} (\|\partial_3^2 u^3\|_{\mathcal{H}_\theta} \|\partial_3 u^3\|_{\mathcal{H}_\theta}^{\frac{2}{p}} \|\nabla_h \partial_3 u^3\|_{\mathcal{H}_\theta}^{1-\frac{2}{p}} + \|\partial_3^2 b^3\|_{\mathcal{H}_\theta} \|\partial_3 b^3\|_{\mathcal{H}_\theta}^{\frac{2}{p}} \|\nabla_h \partial_3 b^3\|_{\mathcal{H}_\theta}^{1-\frac{2}{p}}) \\
& \quad + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} (\|\partial_3^2 b^3\|_{\mathcal{H}_\theta} \|\partial_3 u^3\|_{\mathcal{H}_\theta}^{\frac{2}{p}} \|\nabla_h \partial_3 u^3\|_{\mathcal{H}_\theta}^{1-\frac{2}{p}} + \|\partial_3^2 u^3\|_{\mathcal{H}_\theta} \|\partial_3 b^3\|_{\mathcal{H}_\theta}^{\frac{2}{p}} \|\nabla_h \partial_3 b^3\|_{\mathcal{H}_\theta}^{1-\frac{2}{p}})
\end{aligned} \tag{55}$$

where we used (95), Lemma 2.7 with $p = q = 2, \alpha = \frac{1}{2}, s = \frac{1}{2} + \frac{2}{p}$ and horizontal Gagliardo-Nirenberg inequality. Hence, in sum of (54), (55) in (53) we obtain

$$\begin{aligned}
& (Q_3(u, b)|\partial_3 u^3)_{\mathcal{H}_\theta} + (R_2(u, b)|\partial_3 b^3)_{\mathcal{H}_\theta} \\
& \lesssim \left(\|\omega_{\frac{3}{4}}\|_{L^2}^2 + \|d_{\frac{3}{4}}\|_{L^2}^2 \right)^{\frac{p+6}{6p}} \left(\|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 + \|\nabla d_{\frac{3}{4}}\|_{L^2}^2 \right)^{\frac{1}{2}(1-\frac{2}{p})} \\
& \quad + \left(\|\partial_3 u^3\|_{\mathcal{H}_\theta}^{\frac{2}{p}} + \|\partial_3 b^3\|_{\mathcal{H}_\theta}^{\frac{2}{p}} \right) \left(\|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta}^{1-\frac{2}{p}} + \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta}^{1-\frac{2}{p}} \right) \\
& \quad \times \left(\|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta} + \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta} \right) \left(\|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \right) \\
& \quad + \left(\|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \right) \left(\|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta} + \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta} \right)^{2(1-\frac{1}{p})} \\
& \quad \times \left(\|\partial_3 u^3\|_{\mathcal{H}_\theta} + \|\partial_3 b^3\|_{\mathcal{H}_\theta} \right)^{\frac{2}{p}} \\
& \leq \frac{1}{3} (\|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta}^2 + \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta}^2) \\
& \quad + c \left(\|\omega_{\frac{3}{4}}\|_{L^2}^2 + \|d_{\frac{3}{4}}\|_{L^2}^2 \right)^{\frac{p+6}{3p}} \left(\|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 + \|\nabla d_{\frac{3}{4}}\|_{L^2}^2 \right)^{1-\frac{2}{p}} \left(\|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^2 + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^2 \right) \\
& \quad + \left(\|\partial_3 u^3\|_{\mathcal{H}_\theta}^2 + \|\partial_3 b^3\|_{\mathcal{H}_\theta}^2 \right) \left(\|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p \right)
\end{aligned} \tag{56}$$

by Young's inequalities. Therefore, in sum of (44)-(52), (56) into (42), we obtain

$$\begin{aligned}
& \partial_t (\|\partial_3 u^3\|_{\mathcal{H}_\theta}^2 + \|\partial_3 b^3\|_{\mathcal{H}_\theta}^2) + \|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta}^2 + \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta}^2 \\
& \quad - c (\|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p) (\|\partial_3 u^3\|_{\mathcal{H}_\theta}^2 + \|\partial_3 b^3\|_{\mathcal{H}_\theta}^2) \\
& \lesssim \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^2 (\|\omega_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+3}{3p})} \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^{2(1-\frac{1}{p})} + \|d_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+3}{3p})} \|\nabla d_{\frac{3}{4}}\|_{L^2}^{2(1-\frac{1}{p})}) \\
& \quad + (\|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^2 + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^2) (\|\omega_{\frac{3}{4}}\|_{L^2}^2 + \|d_{\frac{3}{4}}\|_{L^2}^2)^{\frac{p+6}{3p}} (\|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 + \|\nabla d_{\frac{3}{4}}\|_{L^2}^2)^{1-\frac{2}{p}}.
\end{aligned} \tag{57}$$

Gronwall's type argument using (8) and that $\|f\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|\nabla \times f\|_{L^{\frac{3}{2}}}$ for any f such that $\nabla \cdot f = 0$ by Sobolev embedding of $\dot{W}^{\frac{1}{2}, \frac{3}{2}}(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$ and continuity of Riesz transform in $L^{\frac{3}{2}}(\mathbb{R}^3)$ completes the proof of Proposition 3.2. \square

We fix for $T < T^*$

$$e(T) \triangleq c \exp \left(c \int_0^T \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b\|_{L^{\frac{p_1}{p_1-3}}}^{\frac{2p_1}{p_1-3}} + \|\nabla b\|_{L^{\frac{2p_2}{2p_2-3}}}^{\frac{2p_2}{2p_2-3}} d\tau \right). \tag{58}$$

Proposition 3.3. *Under the hypothesis of Theorem 1.1, for $\theta \in (\frac{1}{2} - \frac{2}{p}, \frac{1}{6})$, the solution to the MHD system (1a)-(1c) satisfies for any $t \leq T$*

$$\begin{aligned}
& (\|\omega_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+3}{3p})} + \|d_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+3}{3p})})(t) + \|\nabla \omega_{\frac{3}{4}}\|_{L_t^2 L^2}^{2(\frac{p+3}{3p})} + \|\nabla d_{\frac{3}{4}}\|_{L_t^2 L^2}^{2(\frac{p+3}{3p})} \\
& \lesssim \exp(e(T)) [\|\Omega_0\|_{L^{\frac{3}{2}}}^{\frac{p+3}{2}} + \|j_0\|_{L^{\frac{3}{2}}}^{\frac{p+3}{2}}]
\end{aligned} \tag{59}$$

and

$$\begin{aligned}
& (\|\partial_3 u^3\|_{\mathcal{H}_\theta}^2 + \|\partial_3 b^3\|_{\mathcal{H}_\theta}^2)(t) + \int_0^t \|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta}^2 + \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta}^2 d\tau \\
& \lesssim \exp(e(T)) [\|\Omega_0\|_{L^{\frac{3}{2}}}^2 + \|j_0\|_{L^{\frac{3}{2}}}^2].
\end{aligned} \tag{60}$$

Proof. For $t \leq T$ we let

$$\begin{aligned} III_1(t) &\triangleq \left(\int_0^t \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^2 (\|\omega_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+3}{3p})}) \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^{2(1-\frac{1}{p})} + \|d_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+3}{3p})} \|\nabla d_{\frac{3}{4}}\|_{L^2}^{2(1-\frac{1}{p})} d\tau \right)^{\frac{3}{4}}, \\ III_2(t) &\triangleq \left(\int_0^t (\|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^2 + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^2) (\|\omega_{\frac{3}{4}}\|_{L^2}^2 + \|d_{\frac{3}{4}}\|_{L^2}^2)^{\frac{p+6}{3p}} (\|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 + \|\nabla d_{\frac{3}{4}}\|_{L^2}^2)^{1-\frac{2}{p}} d\tau \right)^{\frac{3}{4}}. \end{aligned}$$

By Proposition 3.2 we have

$$\begin{aligned} e(T) &\left(\int_0^t \|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta}^2 + \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta}^2 d\tau \right)^{\frac{3}{4}} \\ &\leq e(T) \left(\|\Omega_0\|_{L^{\frac{3}{2}}}^{\frac{3}{2}} + \|j_0\|_{L^{\frac{3}{2}}}^{\frac{3}{2}} + III_1(t) + III_2(t) \right) \end{aligned} \quad (61)$$

as $(a+b)^{\frac{3}{4}} \approx a^{\frac{3}{4}} + b^{\frac{3}{4}}$. We estimate

$$\begin{aligned} e(T) III_1(t) &\leq e(T) \left(\int_0^t \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p \|\omega_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+3}{3})} d\tau \right)^{\frac{3}{4p}} \left(\int_0^t \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 d\tau \right)^{\frac{3}{4}(1-\frac{1}{p})} \\ &\quad + e(T) \left(\int_0^t \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p \|d_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+3}{3})} d\tau \right)^{\frac{3}{4p}} \left(\int_0^t \|\nabla d_{\frac{3}{4}}\|_{L^2}^2 d\tau \right)^{\frac{3}{4}(1-\frac{1}{p})} \\ &\leq \frac{1}{9} \int_0^t \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 + \|\nabla d_{\frac{3}{4}}\|_{L^2}^2 d\tau \\ &\quad + e(T) \left(\int_0^t \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p \left(\|\omega_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+3}{3})} + \|d_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+3}{3})} \right) d\tau \right)^{\frac{3}{p+3}} \end{aligned} \quad (62)$$

by Hölder's and Young's inequalities. Similarly,

$$\begin{aligned} e(T) III_2(t) &\leq e(T) \left(\int_0^t (\|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p) (\|\omega_{\frac{3}{4}}\|_{L^2}^2 + \|d_{\frac{3}{4}}\|_{L^2}^2)^{\frac{p+6}{6}} d\tau \right)^{\frac{3}{2p}} \\ &\quad \times \left(\int_0^t \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 + \|\nabla d_{\frac{3}{4}}\|_{L^2}^2 d\tau \right)^{\frac{3}{4}(1-\frac{2}{p})} \\ &\leq \frac{1}{9} \int_0^t \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 + \|\nabla d_{\frac{3}{4}}\|_{L^2}^2 d\tau \\ &\quad + e(T) \left(\int_0^t (\|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p) (\|\omega_{\frac{3}{4}}\|_{L^2}^2 + \|d_{\frac{3}{4}}\|_{L^2}^2)^{\frac{p+6}{6}} d\tau \right)^{\frac{6}{p+6}} \\ &\leq \frac{1}{9} \int_0^t \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 + \|\nabla d_{\frac{3}{4}}\|_{L^2}^2 d\tau \\ &\quad + e(T) \left(\int_0^t (\|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p) (\|\omega_{\frac{3}{4}}\|_{L^2}^2 + \|d_{\frac{3}{4}}\|_{L^2}^2)^{\frac{p+3}{3}} d\tau \right)^{\frac{3}{p+3}} \end{aligned} \quad (63)$$

where we used Hölder's and Young's inequalities and that

$$\begin{aligned} & c \exp \left(c_1 \int_0^T \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b\|_{L^{p_1}}^{\frac{2p_1}{p_1-3}} + \|\nabla b\|_{L^{p_2}}^{\frac{2p_2}{2p_2-3}} d\tau \right) \\ & \times \left(\int_0^t \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p d\tau \right)^{\frac{3p}{(p+3)(p+6)}} \\ & \leq c \exp \left(c_2 \int_0^T \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b\|_{L^{p_1}}^{\frac{2p_1}{p_1-3}} + \|\nabla b\|_{L^{p_2}}^{\frac{2p_2}{2p_2-3}} d\tau \right) \end{aligned}$$

for $c_2 > c_1$ sufficiently large. Therefore, by (62) and (63) applied to (61),

$$\begin{aligned} & e(T) \left(\int_0^t \|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta}^2 + \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta}^2 d\tau \right)^{\frac{3}{4}} \\ & \leq \frac{2}{9} \int_0^t \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 + \|\nabla d_{\frac{3}{4}}\|_{L^2}^2 d\tau \\ & \quad + e(T) [\|\Omega_0\|_{L^{\frac{3}{2}}}^{\frac{3}{2}} + \|j_0\|_{L^{\frac{3}{2}}}^{\frac{3}{2}} \\ & \quad + \left(\int_0^t (\|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p) (\|\omega_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+3}{3})} + \|d_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+3}{3})}) d\tau \right)^{\frac{3}{p+3}}]. \end{aligned} \quad (64)$$

This leads to

$$\begin{aligned} & \frac{2}{3} (\|\omega_{\frac{3}{4}}\|_{L^2}^2 + \|d_{\frac{3}{4}}\|_{L^2}^2)(t) + \frac{5}{9} \int_0^t \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 + \|\nabla d_{\frac{3}{4}}\|_{L^2}^2 d\tau \\ & \leq \frac{2}{9} \int_0^t \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 + \|\nabla d_{\frac{3}{4}}\|_{L^2}^2 d\tau \\ & \quad + e(T) [\|\Omega_0\|_{L^{\frac{3}{2}}}^{\frac{3}{2}} + \|j_0\|_{L^{\frac{3}{2}}}^{\frac{3}{2}} \\ & \quad + \left(\int_0^t (\|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p) (\|\omega_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+3}{3})} + \|d_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+3}{3})}) d\tau \right)^{\frac{3}{p+3}}] \end{aligned} \quad (65)$$

by Proposition 3.1, (64) and that $\|\omega_{\frac{3}{4}}(0)\|_{L^2}^2 = \int |\omega_0|^{\frac{3}{2}} \leq \int |\Omega_0|^{\frac{3}{2}} = \|\Omega_0\|_{L^{\frac{3}{2}}}^{\frac{3}{2}}$. After absorbing, we take powers $\frac{p+3}{3}$ and use $(a+b)^{\frac{p+3}{3}} \approx a^{\frac{p+3}{3}} + b^{\frac{p+3}{3}}$ to obtain

$$\begin{aligned} & \left(\|\omega_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+3}{3})} + \|d_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+3}{3})} \right) (t) + \|\nabla \omega_{\frac{3}{4}}\|_{L_t^2 L^2}^{2(\frac{p+3}{3})} + \|\nabla d_{\frac{3}{4}}\|_{L_t^2 L^2}^{2(\frac{p+3}{3})} \\ & \leq e(T) [\|\Omega_0\|_{L^{\frac{3}{2}}}^{\frac{p+3}{2}} + \|j_0\|_{L^{\frac{3}{2}}}^{\frac{p+3}{2}} \\ & \quad + \int_0^t (\|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p) (\|\omega_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+3}{3})} + \|d_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+3}{3})}) d\tau]. \end{aligned}$$

Thus, Gronwall's type argument using again $e(T) \int_0^t \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p d\tau \lesssim e(T)$ by changing the constant in the exponent leads to (59).

Next, by our Proposition 3.2

$$\begin{aligned}
& (\|\partial_3 u^3\|_{\mathcal{H}_\theta}^2 + \|\partial_3 b^3\|_{\mathcal{H}_\theta}^2)(t) + \int_0^t \|\nabla \partial_3 u^3\|_{\mathcal{H}_\theta}^2 + \|\nabla \partial_3 b^3\|_{\mathcal{H}_\theta}^2 d\tau \\
& \leq e(T) [\|\Omega_0\|_{L^{\frac{3}{2}}}^2 + \|j_0\|_{L^{\frac{3}{2}}}^2 \\
& \quad + \left(\int_0^t \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p d\tau \right)^{\frac{1}{p}} \sup_{\tau \in [0,t]} \left(\|\omega_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+3}{3p})} + \|d_{\frac{3}{4}}\|_{L^2}^{2(\frac{p+3}{3p})} \right) (\tau) \\
& \quad \times \left(\int_0^t \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 + \|\nabla d_{\frac{3}{4}}\|_{L^2}^2 d\tau \right)^{1-\frac{1}{p}} \\
& \quad + \left(\int_0^t \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p d\tau \right)^{\frac{2}{p}} \sup_{\tau \in [0,t]} \left(\|\omega_{\frac{3}{4}}\|_{L^2}^2 + \|d_{\frac{3}{4}}\|_{L^2}^2 \right)^{\frac{p+6}{3p}} (\tau) \\
& \quad \times \left(\int_0^t \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 + \|\nabla d_{\frac{3}{4}}\|_{L^2}^2 d\tau \right)^{1-\frac{2}{p}}] \\
& \leq e(T) [\|\Omega_0\|_{L^{\frac{3}{2}}}^2 + \|j_0\|_{L^{\frac{3}{2}}}^2 \\
& \quad + \|u^3\|_{L_t^p \dot{H}^{\frac{1}{2}+\frac{2}{p}}} \left(e^{e(T)} [\|\Omega_0\|_{L^{\frac{3}{2}}}^{(\frac{p+3}{2})} + \|j_0\|_{L^{\frac{3}{2}}}^{(\frac{p+3}{2})}] \right)^{\frac{1}{p} + \frac{3}{p}(\frac{p-1}{p+3})} \\
& \quad + \left(\|u^3\|_{L_t^p \dot{H}^{\frac{1}{2}+\frac{2}{p}}} + \|b^3\|_{L_t^p \dot{H}^{\frac{1}{2}+\frac{2}{p}}} \right)^2 \left(e^{e(T)} [\|\Omega_0\|_{L^{\frac{3}{2}}}^{(\frac{p+3}{2})} + \|j_0\|_{L^{\frac{3}{2}}}^{(\frac{p+3}{2})}] \right)^{\frac{p+6}{p(p+3)} + \frac{3}{p}(\frac{p-2}{p+3})}] \\
& \lesssim \exp(e(T)) [\|\Omega_0\|_{L^{\frac{3}{2}}}^2 + \|j_0\|_{L^{\frac{3}{2}}}^2]
\end{aligned} \tag{66}$$

by Hölder's inequalities and (59). This completes the proof of Proposition 3.3. \square

4. BLOW-UP CRITERION

Proposition 4.1. *Let $u, b \in C(0, T^*; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)) \cap L^2(0, T^*; \dot{H}^{\frac{3}{2}}(\mathbb{R}^3))$ solve the MHD system (1a)-(1c). If $T^* < \infty$, then for any $p_{k,l} \in (1, \infty)$, $k, l \in \{1, 2, 3\}$,*

$$\sum_{k,l=1}^3 \int_0^{T^*} \|\partial_l u^k\|_{\mathcal{B}_{p_{k,l}}}^{p_{k,l}} + \|\partial_l b^k\|_{\mathcal{B}_{p_{k,l}}}^{p_{k,l}} d\tau = \infty.$$

Proof. We apply $\dot{\Delta}_j$ on (1a)-(1b), take L^2 -inner products with $\dot{\Delta}_j u, \dot{\Delta}_j b$ respectively, sum the two equations, multiplying by 2^j and sum over $j \in \mathbb{Z}$ to obtain

$$\begin{aligned}
& \frac{1}{2} \partial_t (\|u\|_{\dot{H}^{\frac{1}{2}}}^2 + \|b\|_{\dot{H}^{\frac{1}{2}}}^2) + \|\nabla u\|_{\dot{H}^{\frac{1}{2}}}^2 + \|\nabla b\|_{\dot{H}^{\frac{1}{2}}}^2 \\
& = - (u \cdot \nabla u|u)_{\dot{H}^{\frac{1}{2}}} + (b \cdot \nabla b|u)_{\dot{H}^{\frac{1}{2}}} - (u \cdot \nabla b|b)_{\dot{H}^{\frac{1}{2}}} + (b \cdot \nabla u|b)_{\dot{H}^{\frac{1}{2}}}.
\end{aligned} \tag{67}$$

Now we show that

$$\begin{aligned}
& |(u \cdot \nabla u|u)_{\dot{H}^{\frac{1}{2}}} + (u \cdot \nabla b|b)_{\dot{H}^{\frac{1}{2}}} - (b \cdot \nabla b|u)_{\dot{H}^{\frac{1}{2}}} - (b \cdot \nabla u|b)_{\dot{H}^{\frac{1}{2}}}| \\
& \lesssim \sum_{k,l=1}^3 (\|\partial_l u^k\|_{\mathcal{B}_{p_{k,l}}} + \|\partial_l b^k\|_{\mathcal{B}_{p_{k,l}}}) (\|u\|_{\dot{H}^{\frac{1}{2}}}^{\frac{2}{p_{k,l}}} + \|b\|_{\dot{H}^{\frac{1}{2}}}^{\frac{2}{p_{k,l}}}) (\|\nabla u\|_{\dot{H}^{\frac{1}{2}}}^{2(1-\frac{1}{p_{k,l}})} + \|\nabla b\|_{\dot{H}^{\frac{1}{2}}}^{2(1-\frac{1}{p_{k,l}})}).
\end{aligned} \tag{68}$$

Here, making use of the structure of the MHD system becomes important as otherwise, the proof leads us to a non-favorable condition of $p_{l,k} > 2$.

By Lemma 8.1 of [12], we already have

$$|(u \cdot \nabla u)u|_{\dot{H}^{\frac{1}{2}}} \lesssim \sum_{k,l=1}^3 \|\partial_l u^k\|_{\mathcal{B}_{p_k,l}} \|u\|_{\dot{H}^{\frac{1}{2}}}^{\frac{2}{p_{k,l}}} \|\nabla u\|_{\dot{H}^{\frac{1}{2}}}^{2(1-\frac{1}{p_{k,l}})}. \quad (69)$$

Firstly we work on

$$\begin{aligned} & |(u^l \partial_l b^k | b^k)_{\dot{H}^{\frac{1}{2}}}| \\ & \leq \sum_j 2^j |(\dot{\Delta}_j T(u^l, \partial_l b^k) | \dot{\Delta}_j b^k)| + \sum_j 2^j |(\dot{\Delta}_j T(\partial_l b^k, u^l) | \dot{\Delta}_j b^k)| \\ & \quad + \sum_j 2^j |(\dot{\Delta}_j R(u^l, \partial_l b^k) | \dot{\Delta}_j b^k)| \triangleq IV_1 + IV_2 + IV_3 \end{aligned} \quad (70)$$

due to Bony's paraproduct decomposition (6). We start with

$$\begin{aligned} IV_1 & \leq \sum_j 2^j \left| \int \dot{S}_{j-1} u^l \dot{\Delta}_j \partial_l b^k \dot{\Delta}_j b^k \right| + \left| \int \sum_{|j-j'|\leq 4} [\dot{\Delta}_j, \dot{S}_{j'-1} u^l] \dot{\Delta}_{j'} \partial_l b^k \dot{\Delta}_j b^k \right| \\ & \quad + \left| \int \sum_{|j-j'|\leq 4} (\dot{S}_{j'-1} u^l - \dot{S}_{j-1} u^l) \dot{\Delta}_j \dot{\Delta}_{j'} \partial_l b^k \dot{\Delta}_j b^k \right| \triangleq IV_{1,1} + IV_{1,2} + IV_{1,3}. \end{aligned} \quad (71)$$

We have due to divergence-free property, $IV_{1,1} = 0$. Secondly,

$$\begin{aligned} IV_{1,2} & \leq \sum_j 2^j \sum_{|j-j'|\leq 4} \|[\dot{\Delta}_j, \dot{S}_{j'-1} u^l] \dot{\Delta}_{j'} \partial_l b^k\|_{L^2} \|\dot{\Delta}_j b^k\|_{L^2} \\ & \lesssim \sum_j 2^j 2^{-j} \sum_{|j-j'|\leq 4} \|\nabla \dot{S}_{j'-1} u^l\|_{L^\infty} \|\dot{\Delta}_{j'} \partial_l b^k\|_{L^2} \|\dot{\Delta}_j b^k\|_{L^2} \\ & \lesssim \sum_j \sum_{l'=1}^3 \|\partial_{l'} \dot{S}_{j-1} u^l\|_{L^\infty} \|\dot{\Delta}_j \partial_l b^k\|_{L^2} \|\dot{\Delta}_j b^k\|_{L^2} \\ & \lesssim \sum_{l'=1}^3 \sum_j 2^{j(2-\frac{2}{p_{l,l'}})} \\ & \quad \times \sum_{j' \leq j-2} 2^{(j'-j)(2-\frac{2}{p_{l,l'}})} 2^{j'(-2+\frac{2}{p_{l,l'}})} \|\dot{\Delta}_{j'} \partial_{l'} u^l\|_{L^\infty} \|\dot{\Delta}_j \partial_l b^k\|_{L^2} \|\dot{\Delta}_j b^k\|_{L^2} \\ & \lesssim \sum_{l'=1}^3 \|\partial_{l'} u^l\|_{\mathcal{B}_{p_{l,l'}}} \sum_j (2^{-\frac{j}{2}} \|\dot{\Delta}_j \partial_l b^k\|_{L^2})^{\frac{1}{p_{l,l'}}} (2^{\frac{j}{2}} \|\dot{\Delta}_j b^k\|_{L^2})^{\frac{1}{p_{l,l'}}} \\ & \quad \times (2^{\frac{j}{2}} \|\dot{\Delta}_j \partial_l b^k\|_{L^2})^{1-\frac{1}{p_{l,l'}}} (2^{\frac{3j}{2}} \|\dot{\Delta}_j b^k\|_{L^2})^{1-\frac{1}{p_{l,l'}}} \\ & \lesssim \sum_{l'=1}^3 \|\partial_{l'} u^l\|_{\mathcal{B}_{p_{l,l'}}} \|b\|_{\dot{H}^{\frac{1}{2}}}^{\frac{2}{p_{l,l'}}} \|\nabla b\|_{\dot{H}^{\frac{1}{2}}}^{2(1-\frac{1}{p_{l,l'}})} \end{aligned} \quad (72)$$

where we used Hölder's inequality, a commutator estimate (cf. Lemma 2.97 [1] and also [28]) and Young's inequality for convolution. Thirdly,

$$\begin{aligned}
IV_{1,3} &\leq \sum_j 2^j \sum_{|j-j'|\leq 4} \|(\dot{S}_{j'-1}u^l - \dot{S}_{j-1}u^l)\dot{\Delta}_j\dot{\Delta}_{j'}\partial_l b^k\|_{L^2} \|\dot{\Delta}_j b^k\|_{L^2} \\
&\lesssim \sum_j 2^j \sum_{|j-j'|\leq 4, j''\in[j-1, j'-1]} \left\| \sum_{l'=1}^3 2^{-j''} \tilde{\Delta}_{j''}^{l'} \dot{\Delta}_{j''} \partial_{l'} u^l \right\|_{L^\infty} \|\dot{\Delta}_j \dot{\Delta}_{j'} \partial_l b^k\|_{L^2} \|\dot{\Delta}_j b^k\|_{L^2} \\
&\lesssim \sum_j 2^j \sum_{l'=1}^3 2^{-j} \|\dot{\Delta}_j \partial_{l'} u^l\|_{L^\infty} \|\dot{\Delta}_j \partial_l b^k\|_{L^2} \|\dot{\Delta}_j b^k\|_{L^2} \\
&\lesssim \sum_{l'=1}^3 \|\partial_{l'} u^l\|_{\mathcal{B}_{p_l, l'}} \sum_j (2^{-\frac{j}{2}} \|\dot{\Delta}_j \partial_l b^k\|_{L^2})^{\frac{1}{p_{l, l'}}} (2^{\frac{j}{2}} \|\dot{\Delta}_j b^k\|_{L^2})^{\frac{1}{p_{l, l'}}} \\
&\quad \times (2^{\frac{j}{2}} \|\dot{\Delta}_j \partial_l b^k\|_{L^2})^{1-\frac{1}{p_{l, l'}}} (2^{\frac{3j}{2}} \|\dot{\Delta}_j b^k\|_{L^2})^{1-\frac{1}{p_{l, l'}}} \\
&\lesssim \sum_{l'=1}^3 \|\partial_{l'} u^l\|_{\mathcal{B}_{p_l, l'}} \|b\|_{\dot{H}^{\frac{1}{2}}}^{\frac{2}{p_{l, l'}}} \|\nabla b\|_{\dot{H}^{\frac{1}{2}}}^{2(1-\frac{1}{p_{l, l'}})}
\end{aligned} \tag{73}$$

by Hölder's inequality; we also used the fact that $|j-j'|\leq 4$ and $j''\in[j-1, j'-1]$ imply that we can assume these indices are all j modifying constants. Therefore, (72), (73) in (71) imply

$$IV_1 \lesssim \sum_{l'=1}^3 \|\partial_{l'} u^l\|_{\mathcal{B}_{p_l, l'}} \|b\|_{\dot{H}^{\frac{1}{2}}}^{\frac{2}{p_{l, l'}}} \|\nabla b\|_{\dot{H}^{\frac{1}{2}}}^{2(1-\frac{1}{p_{l, l'}})}. \tag{74}$$

Next,

$$\begin{aligned}
IV_2 &\lesssim \sum_j \sum_{|j-j'|\leq 4} 2^j \|\dot{S}_{j'-1} \partial_l b^k\|_{L^\infty} \|\dot{\Delta}_{j'} u^l\|_{L^2} \|\dot{\Delta}_j b^k\|_{L^2} \\
&\lesssim \sum_j 2^j \|\dot{S}_{j-1} \partial_l b^k\|_{L^\infty} \|\dot{\Delta}_j u^l\|_{L^2} \|\dot{\Delta}_j b^k\|_{L^2} \\
&\lesssim \sum_j 2^{j(3-\frac{2}{p_{k, l}})} \sum_{j'\leq j-2} 2^{(j'-j)(2-\frac{2}{p_{k, l}})} 2^{j'(-2+\frac{2}{p_{k, l}})} \|\dot{\Delta}_{j'} \partial_l b^k\|_{L^\infty} \|\dot{\Delta}_j u^l\|_{L^2} \|\dot{\Delta}_j b^k\|_{L^2} \\
&\lesssim \|\partial_l b^k\|_{\mathcal{B}_{p_{k, l}}} \sum_j (2^{\frac{j}{2}} \|\dot{\Delta}_j u^l\|_{L^2})^{\frac{1}{p_{k, l}}} (2^{\frac{j}{2}} \|\dot{\Delta}_j b^k\|_{L^2})^{\frac{1}{p_{k, l}}} \\
&\quad \times (2^{\frac{3j}{2}} \|\dot{\Delta}_j u^l\|_{L^2})^{1-\frac{1}{p_{k, l}}} (2^{\frac{3j}{2}} \|\dot{\Delta}_j b^k\|_{L^2})^{1-\frac{1}{p_{k, l}}} \\
&\lesssim \|\partial_l b^k\|_{\mathcal{B}_{p_{k, l}}} (\|u\|_{\dot{H}^{\frac{2}{p_{k, l}}}}^{\frac{2}{p_{k, l}}} + \|b\|_{\dot{H}^{\frac{2}{p_{k, l}}}}^{\frac{2}{p_{k, l}}}) (\|\nabla u\|_{\dot{H}^{\frac{1}{2}}}^{2(1-\frac{1}{p_{k, l}})} + \|\nabla b\|_{\dot{H}^{\frac{1}{2}}}^{2(1-\frac{1}{p_{k, l}})})
\end{aligned} \tag{75}$$

where we used Hölder's and Young's inequality for convolution. Finally, we first write by divergence-free condition,

$$IV_3 = \sum_j 2^j |(\dot{\Delta}_j R(u^l, \partial_l b^k) | \dot{\Delta}_j b^k)| = \sum_j 2^j |(\dot{\Delta}_j \sum_{j'\geq j-\delta} \partial_l (\dot{\Delta}_{j'} u^l \tilde{\Delta}_{j'} b^k) | \dot{\Delta}_j b^k)|$$

for some $\delta \in \mathbb{Z}^+0$ and $\dot{\Delta}_{j'} u^l \approx \sum_{l'=1}^3 \tilde{\Delta}_{j'}^{l'} \dot{\Delta}_{j'} u^l \approx \sum_{l'=1}^3 2^{-j'} \tilde{\Delta}_{j'}^{l'} \dot{\Delta}_{j'} \partial_{l'} u^l$ so that

$$\begin{aligned}
IV_3 &\lesssim \sum_j 2^j \|\dot{\Delta}_j \sum_{j' \geq j-\delta} \partial_{l'} (\sum_{l'=1}^3 2^{-j'} \tilde{\Delta}_{j'}^{l'} \dot{\Delta}_{j'} \partial_{l'} u^l \tilde{\Delta}_{j'} b^k)\|_{L^2} \|\dot{\Delta}_j b^k\|_{L^2} \\
&\lesssim \sum_j 2^j \sum_{j' \geq j-\delta} \sum_{l'=1}^3 2^{j-j'} \|\tilde{\Delta}_{j'}^{l'} \dot{\Delta}_{j'} \partial_{l'} u^l\|_{L^\infty} \|\tilde{\Delta}_{j'} b^k\|_{L^2} \|\dot{\Delta}_j b^k\|_{L^2} \\
&\lesssim \sum_{l'=1}^3 \|\partial_{l'} u^l\|_{\mathcal{B}_{p_l, l'}} \sum_j \sum_{j' \geq j-\delta} 2^{(j-j')(\frac{1}{2} + \frac{1}{p_{l, l'}})} \\
&\quad \times (2^{\frac{j'}{2}} \|\tilde{\Delta}_{j'} b\|_{L^2})^{\frac{1}{p_{l, l'}}} (2^{\frac{3j'}{2}} \|\tilde{\Delta}_{j'} b\|_{L^2})^{1-\frac{1}{p_{l, l'}}} (2^{\frac{j}{2}} \|\dot{\Delta}_j b\|_{L^2})^{\frac{1}{p_{l, l'}}} (2^{\frac{3j}{2}} \|\dot{\Delta}_j b\|_{L^2})^{1-\frac{1}{p_{l, l'}}} \\
&\lesssim \sum_{l'=1}^3 \|\partial_{l'} u^l\|_{\mathcal{B}_{p_l, l'}} \|(2^{\frac{j}{2}} \|\tilde{\Delta}_j b\|_{L^2})^{\frac{1}{p_{l, l'}}} (2^{\frac{3j}{2}} \|\tilde{\Delta}_j b\|_{L^2})^{1-\frac{1}{p_{l, l'}}}\|_{l^2} \\
&\quad \times \|(2^{\frac{j}{2}} \|\dot{\Delta}_j b\|_{L^2})^{\frac{1}{p_{l, l'}}} (2^{\frac{3j}{2}} \|\dot{\Delta}_j b\|_{L^2})^{1-\frac{1}{p_{l, l'}}}\|_{l^2} \\
&\lesssim \sum_{l'=1}^3 \|\partial_{l'} u^l\|_{\mathcal{B}_{p_l, l'}} \|b\|_{\dot{H}^{\frac{1}{2}}}^{\frac{2}{p_{l, l'}}} \|\nabla b\|_{\dot{H}^{\frac{1}{2}}}^{2(1-\frac{1}{p_{l, l'}})}
\end{aligned} \tag{76}$$

by Hölder's, Bernstein's and Young's inequality for convolution. Thus, from (74)-(76) applied to (70)

$$\begin{aligned}
&|(u^l \partial_l b^k | b^k)_{\dot{H}^{\frac{1}{2}}}| \\
&\lesssim \sum_{k, l=1}^3 (\|\partial_l u^k\|_{\mathcal{B}_{p_k, l}} + \|\partial_l b^k\|_{\mathcal{B}_{p_k, l}}) (\|u\|_{\dot{H}^{\frac{1}{2}}}^{\frac{2}{p_{k, l}}} + \|b\|_{\dot{H}^{\frac{1}{2}}}^{\frac{2}{p_{k, l}}}) (\|\nabla u\|_{\dot{H}^{\frac{1}{2}}}^{2(1-\frac{1}{p_{k, l}})} + \|\nabla b\|_{\dot{H}^{\frac{1}{2}}}^{2(1-\frac{1}{p_{k, l}})}).
\end{aligned} \tag{77}$$

Next, we work on

$$\begin{aligned}
(b^l \partial_l b^k | u^k)_{\dot{H}^{\frac{1}{2}}} &= \sum_j 2^j (\dot{\Delta}_j T(b^l, \partial_l b^k) | \dot{\Delta}_j u^k) + \sum_j 2^j (\dot{\Delta}_j T(\partial_l b^k, b^l) | \dot{\Delta}_j u^k) \\
&\quad + \sum_j 2^j (\dot{\Delta}_j R(b^l, \partial_l b^k) | \dot{\Delta}_j u^k) \triangleq V_1 + V_2 + V_3
\end{aligned} \tag{78}$$

due to Bony's paraproduct decomposition (6). Let us work on V_1 subsequently together with VI_1 to be defined below. We now estimate similarly to IV_2 in (75)

$$\begin{aligned}
V_2 &\lesssim \sum_j 2^j \|\dot{S}_{j-1} \partial_l b^k\|_{L^\infty} \|\dot{\Delta}_j b^l\|_{L^2} \|\dot{\Delta}_j u^k\|_{L^2} \\
&\lesssim \|\partial_l b^k\|_{\mathcal{B}_{p_k, l}} \sum_j (2^{\frac{j}{2}} \|\dot{\Delta}_j b^l\|_{L^2})^{\frac{1}{p_{k, l}}} (2^{\frac{j}{2}} \|\dot{\Delta}_j u^k\|_{L^2})^{\frac{1}{p_{k, l}}} \\
&\quad \times (2^{\frac{3j}{2}} \|\dot{\Delta}_j b^l\|_{L^2})^{1-\frac{1}{p_{k, l}}} (2^{\frac{3j}{2}} \|\dot{\Delta}_j u^k\|_{L^2})^{1-\frac{1}{p_{k, l}}} \\
&\lesssim \|\partial_l b^k\|_{\mathcal{B}_{p_k, l}} (\|b\|_{\dot{H}^{\frac{1}{2}}}^{\frac{2}{p_{k, l}}} + \|u\|_{\dot{H}^{\frac{1}{2}}}^{\frac{2}{p_{k, l}}}) (\|\nabla b\|_{\dot{H}^{\frac{1}{2}}}^{2(1-\frac{1}{p_{k, l}})} + \|\nabla u\|_{\dot{H}^{\frac{1}{2}}}^{2(1-\frac{1}{p_{k, l}})})
\end{aligned} \tag{79}$$

by Hölder's and Young's inequality for convolution. Next, as done in (76), by divergence-free condition, and writing $\dot{\Delta}_{j'} b^l \approx \sum_{l'=1}^3 \tilde{\Delta}_{j'}^{l'} \dot{\Delta}_{j'} b^l \approx \sum_{l'=1}^3 2^{-j'} \tilde{\Delta}_{j'}^{l'} \dot{\Delta}_{j'} \partial_{l'} b^l$,

we estimate

$$\begin{aligned}
V_3 &\approx \sum_j 2^j (\dot{\Delta}_j \sum_{j' \geq j-\delta} \partial_l ((\sum_{l'=1}^3 2^{-j'} \tilde{\Delta}_{j'} \dot{\Delta}_{j'} \partial_{l'} b^l) \tilde{\Delta}_{j'} b^k) | \dot{\Delta}_j b^k) \\
&\lesssim \sum_j 2^j \sum_{j' \geq j-\delta} \sum_{l'=1}^3 2^{j-j'} \|\tilde{\Delta}_{j'} \dot{\Delta}_{j'} \partial_{l'} b^l\|_{L^\infty} \|\tilde{\Delta}_{j'} b^k\|_{L^2} \|\dot{\Delta}_j b^k\|_{L^2} \\
&\lesssim \sum_{l'=1}^3 \|\partial_{l'} b^l\|_{\mathcal{B}_{p_l, l'}} \sum_j \sum_{j' \geq j-\delta} 2^{(j-j')(\frac{1}{2} + \frac{1}{p_{l, l'}})} \\
&\quad \times (2^{\frac{j'}{2}} \|\tilde{\Delta}_{j'} b^k\|_{L^2})^{\frac{1}{p_{l, l'}}} (2^{\frac{3j'}{2}} \|\tilde{\Delta}_{j'} b^k\|_{L^2})^{1 - \frac{1}{p_{l, l'}}} (2^{\frac{j}{2}} \|\dot{\Delta}_j b^k\|_{L^2})^{\frac{1}{p_{l, l'}}} (2^{\frac{3j}{2}} \|\dot{\Delta}_j b^k\|_{L^2})^{1 - \frac{1}{p_{l, l'}}} \\
&\lesssim \sum_{l'=1}^3 \|\partial_{l'} b^l\|_{\mathcal{B}_{p_l, l'}} \|(2^{\frac{j}{2}} \|\dot{\Delta}_j b^k\|_{L^2})^{\frac{1}{p_{l, l'}}} (2^{\frac{3j}{2}} \|\dot{\Delta}_j b^k\|_{L^2})^{1 - \frac{1}{p_{l, l'}}}\|_{l_2^2}^2 \\
&\lesssim \sum_{l'=1}^3 \|\partial_{l'} b^l\|_{\mathcal{B}_{p_l, l'}} \|b\|_{\dot{H}^{\frac{1}{2}}}^{\frac{2}{p_{l, l'}}} \|\nabla b\|_{\dot{H}^{\frac{1}{2}}}^{2(1 - \frac{1}{p_{l, l'}})}
\end{aligned} \tag{80}$$

by Hölder's, Bernstein's and Young's inequality for convolution.

Finally, we consider

$$\begin{aligned}
(b^l \partial_l u^k | b^k)_{\dot{H}^{\frac{1}{2}}} &= \sum_j 2^j (\dot{\Delta}_j T(b^l, \partial_l u^k) | \dot{\Delta}_j b^k) + \sum_j 2^j (\dot{\Delta}_j T(\partial_l u^k, b^l) | \dot{\Delta}_j b^k) \\
&\quad + \sum_j 2^j (\dot{\Delta}_j R(b^l, \partial_l u^k) | \dot{\Delta}_j b^k) \triangleq VI_1 + VI_2 + VI_3.
\end{aligned} \tag{81}$$

We now consider V_1 from (78) along with VI_1 of (81):

$$\begin{aligned}
V_1 + VI_1 &= \sum_j 2^j \int \dot{S}_{j-1} b^l \dot{\Delta}_j \partial_l b^k \dot{\Delta}_j u^k + \sum_{|j-j'| \leq 4} [\dot{\Delta}_j, \dot{S}_{j'-1} b^l] \dot{\Delta}_{j'} \partial_l b^k \dot{\Delta}_j u^k \\
&\quad + \sum_{|j-j'| \leq 4} (\dot{S}_{j'-1} b^l - \dot{S}_{j-1} b^l) \dot{\Delta}_j \dot{\Delta}_{j'} \partial_l b^k \dot{\Delta}_j u^k + \dot{S}_{j-1} b^l \dot{\Delta}_j \partial_l u^k \dot{\Delta}_j b^k \\
&\quad + \sum_{|j-j'| \leq 4} [\dot{\Delta}_j, \dot{S}_{j'-1} b^l] \dot{\Delta}_{j'} \partial_l u^k \dot{\Delta}_j b^k \\
&\quad + \sum_{|j-j'| \leq 4} (\dot{S}_{j'-1} b^l - \dot{S}_{j-1} b^l) \dot{\Delta}_j \dot{\Delta}_{j'} \partial_l u^k \dot{\Delta}_j b^k \triangleq \sum_{i=1}^6 (V_1 + VI_1)_i.
\end{aligned} \tag{82}$$

We make use of that, together due to the divergence-free property of b , $(V_1 + VI_1)_1 + (V_1 + VI_1)_4 = 0$. Now we work similarly to (72) on

$$\begin{aligned}
& (V_1 + VI_1)_2 + (V_1 + VI_1)_5 \tag{83} \\
& \lesssim \sum_j 2^j 2^{-j} \sum_{|j-j'|\leq 4} \|\nabla \dot{S}_{j'-1} b^l\|_{L^\infty} (\|\dot{\Delta}_{j'} \partial_l b^k\|_{L^2} \|\dot{\Delta}_j u^k\|_{L^2} + \|\dot{\Delta}_{j'} \partial_l u^k\|_{L^2} \|\dot{\Delta}_j b^k\|_{L^2}) \\
& \lesssim \sum_{l'=1}^3 \sum_j \|\partial_{l'} \dot{S}_{j-1} b^l\|_{L^\infty} (\|\dot{\Delta}_j \partial_l b^k\|_{L^2} \|\dot{\Delta}_j u^k\|_{L^2} + \|\dot{\Delta}_j \partial_l u^k\|_{L^2} \|\dot{\Delta}_j b^k\|_{L^2}) \\
& \lesssim \sum_{l'=1}^3 \sum_j 2^{j(2-\frac{2}{p_{l,l'}})} \sum_{j' \leq j-2} 2^{(j'-j)(2-\frac{2}{p_{l,l'}})} 2^{j'(-2+\frac{2}{p_{l,l'}})} \|\dot{\Delta}_{j'} \partial_{l'} b^l\|_{L^\infty} \\
& \quad \times (\|\dot{\Delta}_j \partial_l b^k\|_{L^2} \|\dot{\Delta}_j u^k\|_{L^2} + \|\dot{\Delta}_j \partial_l u^k\|_{L^2} \|\dot{\Delta}_j b^k\|_{L^2}) \\
& \lesssim \sum_{l'=1}^3 \|\partial_{l'} b^l\|_{\mathcal{B}_{p_{l,l'}}} \sum_j (2^{-\frac{j}{2}} \|\dot{\Delta}_j \partial_l b^k\|_{L^2})^{\frac{1}{p_{l,l'}}} (2^{\frac{j}{2}} \|\dot{\Delta}_j u^k\|_{L^2})^{\frac{1}{p_{l,l'}}} \\
& \quad \times (2^{\frac{j}{2}} \|\dot{\Delta}_j \partial_l b^k\|_{L^2})^{1-\frac{1}{p_{l,l'}}} (2^{\frac{3j}{2}} \|\dot{\Delta}_j u^k\|_{L^2})^{1-\frac{1}{p_{l,l'}}} \\
& \quad + (2^{-\frac{j}{2}} \|\dot{\Delta}_j \partial_l u^k\|_{L^2})^{\frac{1}{p_{l,l'}}} (2^{\frac{j}{2}} \|\dot{\Delta}_j b^k\|_{L^2})^{\frac{1}{p_{l,l'}}} \\
& \quad \times (2^{\frac{j}{2}} \|\dot{\Delta}_j \partial_l u^k\|_{L^2})^{1-\frac{1}{p_{l,l'}}} (2^{\frac{3j}{2}} \|\dot{\Delta}_j b^k\|_{L^2})^{1-\frac{1}{p_{l,l'}}} \\
& \lesssim \sum_{l'=1}^3 \|\partial_{l'} b^l\|_{\mathcal{B}_{p_{l,l'}}} (\|b\|_{\dot{H}^{\frac{1}{2}}}^{\frac{2}{p_{l,l'}}} + \|u\|_{\dot{H}^{\frac{1}{2}}}^{\frac{2}{p_{l,l'}}}) (\|\nabla b\|_{\dot{H}^{\frac{1}{2}}}^{2(1-\frac{1}{p_{l,l'}})} + \|\nabla u\|_{\dot{H}^{\frac{1}{2}}}^{2(1-\frac{1}{p_{l,l'}})})
\end{aligned}$$

by Hölder's inequality, commutator estimate used in (72) and Young's inequality for convolution. Next, we work similarly to $IV_{1,3}$ in (73):

$$\begin{aligned}
& (V_1 + VI_1)_3 + (V_1 + VI_1)_6 \tag{84} \\
& \lesssim \sum_j 2^j \sum_{|j-j'|\leq 4, j'' \in [j-1, j'-1]} \|\sum_{l'=1}^3 2^{-j''} \tilde{\Delta}_{j''}^{l'} \dot{\Delta}_{j''} \partial_{l'} b^l\|_{L^\infty} \\
& \quad \times (\|\dot{\Delta}_j \dot{\Delta}_{j'} \partial_l b^k\|_{L^2} \|\dot{\Delta}_j u^k\|_{L^2} + \|\dot{\Delta}_j \dot{\Delta}_{j'} \partial_l u^k\|_{L^2} \|\dot{\Delta}_j b^k\|_{L^2}) \\
& \lesssim \sum_{l'=1}^3 \|\partial_{l'} b^l\|_{\mathcal{B}_{p_{l,l'}}} \sum_j (2^{-\frac{j}{2}} \|\dot{\Delta}_j \partial_l b^k\|_{L^2})^{\frac{1}{p_{l,l'}}} (2^{\frac{j}{2}} \|\dot{\Delta}_j u^k\|_{L^2})^{\frac{1}{p_{l,l'}}} \\
& \quad \times (2^{\frac{j}{2}} \|\dot{\Delta}_j \partial_l b^k\|_{L^2})^{1-\frac{1}{p_{l,l'}}} (2^{\frac{3j}{2}} \|\dot{\Delta}_j u^k\|_{L^2})^{1-\frac{1}{p_{l,l'}}} \\
& \quad + (2^{-\frac{j}{2}} \|\dot{\Delta}_j \partial_l u^k\|_{L^2})^{\frac{1}{p_{l,l'}}} (2^{\frac{j}{2}} \|\dot{\Delta}_j b^k\|_{L^2})^{\frac{1}{p_{l,l'}}} \\
& \quad \times (2^{\frac{j}{2}} \|\dot{\Delta}_j \partial_l u^k\|_{L^2})^{1-\frac{1}{p_{l,l'}}} (2^{\frac{3j}{2}} \|\dot{\Delta}_j b^k\|_{L^2})^{1-\frac{1}{p_{l,l'}}} \\
& \lesssim \sum_{l'=1}^3 \|\partial_{l'} b^l\|_{\mathcal{B}_{p_{l,l'}}} (\|b\|_{\dot{H}^{\frac{1}{2}}}^{\frac{2}{p_{l,l'}}} + \|u\|_{\dot{H}^{\frac{1}{2}}}^{\frac{2}{p_{l,l'}}}) (\|\nabla b\|_{\dot{H}^{\frac{1}{2}}}^{2(1-\frac{1}{p_{l,l'}})} + \|\nabla u\|_{\dot{H}^{\frac{1}{2}}}^{2(1-\frac{1}{p_{l,l'}})})
\end{aligned}$$

by Hölder's inequality, that we can write $\dot{\Delta}_{j''} b^l \approx \sum_{l'=1}^3 2^{-j''} \tilde{\Delta}_{j''}^{l'} \dot{\Delta}_{j''} \partial_{l'} b^l$ and Young's inequality. Next, we work similarly to IV_2 in (75) to estimate

$$\begin{aligned} VI_2 &\lesssim \sum_j \sum_{|j-j'|\leq 4} 2^j \|\dot{S}_{j'-1} \partial_l u^k\|_{L^\infty} \|\Delta_{j'} b^l\|_{L^2} \|\dot{\Delta}_j b^k\|_{L^2} \\ &\lesssim \sum_j 2^{j(3-\frac{2}{p_{k,l}})} \sum_{j'\leq j-2} 2^{(j'-j)(2-\frac{2}{p_{k,l}})} 2^{j'(-2+\frac{2}{p_{k,l}})} \|\dot{\Delta}_{j'} \partial_l u^k\|_{L^\infty} \|\dot{\Delta}_j b^l\|_{L^2} \|\dot{\Delta}_j b^k\|_{L^2} \\ &\lesssim \|\partial_l u^k\|_{\mathcal{B}_{p_{k,l}}} \|b\|_{\dot{H}^{\frac{1}{2}}}^{\frac{2}{p_{k,l}}} \|\nabla b\|_{\dot{H}^{\frac{1}{2}}}^{2(1-\frac{1}{p_{k,l}})} \end{aligned} \quad (85)$$

by Hölder's inequality and Young's inequality for convolution. Finally, we use divergence-free condition and write $\dot{\Delta}_{j'} b^l \approx \sum_{l'=1}^3 \tilde{\Delta}_{j'}^{l'} \dot{\Delta}_{j'} b^l \approx \sum_{l'=1}^3 2^{-j'} \tilde{\Delta}_{j'}^{l'} \dot{\Delta}_{j'} \partial_{l'} b_l$ so that similarly to IV_3 in (76), we can estimate

$$\begin{aligned} VI_3 &\approx \sum_j 2^j (\dot{\Delta}_j \sum_{j'\geq j-\delta} \partial_l (\sum_{l'=1}^3 2^{-j'} \tilde{\Delta}_{j'}^{l'} \dot{\Delta}_{j'} \partial_{l'} b^l \tilde{\Delta}_{j'} u^k) | \dot{\Delta}_j b^k) \\ &\lesssim \sum_{l'=1}^3 \|\partial_{l'} b^l\|_{\mathcal{B}_{p_{l,l'}}} \sum_j \sum_{j'\geq j-\delta} 2^{(j-j')(\frac{1}{2}+\frac{1}{p_{l,l'}})} (2^{\frac{j'}{2}} \|\tilde{\Delta}_{j'} u^k\|_{L^2})^{\frac{1}{p_{l,l'}}} \\ &\quad \times (2^{\frac{3j'}{2}} \|\tilde{\Delta}_{j'} u^k\|_{L^2})^{1-\frac{1}{p_{l,l'}}} (2^{\frac{j}{2}} \|\dot{\Delta}_j b^k\|_{L^2})^{\frac{1}{p_{l,l'}}} (2^{\frac{3j}{2}} \|\dot{\Delta}_j b^k\|_{L^2})^{1-\frac{1}{p_{l,l'}}} \\ &\lesssim \sum_{l'=1}^3 \|\partial_{l'} b^l\|_{\mathcal{B}_{p_{l,l'}}} \|(2^{\frac{j}{2}} \|\tilde{\Delta}_{j'} u^k\|_{L^2})^{\frac{1}{p_{l,l'}}} (2^{\frac{3j}{2}} \|\tilde{\Delta}_{j'} u^k\|_{L^2})^{1-\frac{1}{p_{l,l'}}}\|_{l^2} \\ &\quad \times \|(2^{\frac{j}{2}} \|\dot{\Delta}_j b^k\|_{L^2})^{\frac{1}{p_{l,l'}}} (2^{\frac{3j}{2}} \|\dot{\Delta}_j b^k\|_{L^2})^{1-\frac{1}{p_{l,l'}}}\|_{l^2} \\ &\lesssim \sum_{l'=1}^3 \|\partial_{l'} b^l\|_{\mathcal{B}_{p_{l,l'}}} (\|u\|_{\dot{H}^{\frac{1}{2}}}^{\frac{2}{p_{l,l'}}} + \|b\|_{\dot{H}^{\frac{1}{2}}}^{\frac{2}{p_{l,l'}}}) (\|\nabla u\|_{\dot{H}^{\frac{1}{2}}}^{2(1-\frac{1}{p_{l,l'}})} + \|\nabla b\|_{\dot{H}^{\frac{1}{2}}}^{2(1-\frac{1}{p_{l,l'}})}) \end{aligned} \quad (86)$$

by Hölder's, Bernstein's and Young's inequality of convolution.

Hence, we obtain (68) due to (69), (77), (79), (80), (82)-(86). Applying (68) to (67), Young's and Gronwall's inequalities complete the proof of the Proposition 4.1. \square

5. PROOF OF (4) IN THEOREM 1.1

Firstly, for any $p \in (4, 6)$,

$$\begin{aligned} \max_{1\leq l\leq 3} (\|\partial_l u^3\|_{\mathcal{B}_p} + \|\partial_l b^3\|_{\mathcal{B}_p}) &\lesssim \sup_{j\in\mathbb{Z}} 2^{j(\frac{1}{2}+\frac{2}{p})} (\|\dot{\Delta}_j u^3\|_{L^2} + \|\dot{\Delta}_j b^3\|_{L^2}) \\ &\lesssim \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \end{aligned}$$

by Bernstein's inequality, which implies

$$\max_{1\leq l\leq 3} \int_0^{T^*} \|\partial_l u^3\|_{\mathcal{B}_p}^p + \|\partial_l b^3\|_{\mathcal{B}_p}^p d\tau \lesssim \int_0^{T^*} \|u^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p + \|b^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p d\tau \lesssim 1. \quad (87)$$

Next, for $T < T^*$ by (5) for any $p \in (4, 6)$

$$\begin{aligned} & \int_0^T \|\nabla_h u^h\|_{\mathcal{B}_p}^p + \|\nabla_h b^h\|_{\mathcal{B}_p}^p d\tau \\ & \approx \int_0^T \|\nabla_h \nabla_h^\perp \Delta_h^{-1} \omega\|_{\mathcal{B}_p}^p + \|\nabla_h \nabla_h^\perp \Delta_h^{-1} d\|_{\mathcal{B}_p}^p \\ & \quad + \|\nabla_h \nabla_h \Delta_h^{-1} \partial_3 u^3\|_{\mathcal{B}_p}^p + \|\nabla_h \nabla_h \Delta_h^{-1} \partial_3 b^3\|_{\mathcal{B}_p}^p d\tau \end{aligned} \quad (88)$$

where

$$\begin{aligned} & \int_0^T \|\nabla_h \nabla_h^\perp \Delta_h^{-1} \omega\|_{\mathcal{B}_p}^p + \|\nabla_h \nabla_h^\perp \Delta_h^{-1} d\|_{\mathcal{B}_p}^p d\tau \\ & \lesssim \int_0^T \|\omega\|_{L^{\frac{3}{2}}}^{p(1-\frac{3}{2p})} \|\nabla \omega\|_{L^{\frac{9}{5}}}^{\frac{3}{2}} + \|d\|_{L^{\frac{3}{2}}}^{p(1-\frac{3}{2p})} \|\nabla d\|_{L^{\frac{9}{5}}}^{\frac{3}{2}} d\tau \\ & \lesssim \int_0^T \|\omega\|_{L^{\frac{3}{2}}}^{p(1-\frac{3}{2p})} \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 + \|d\|_{L^{\frac{3}{2}}}^{p(1-\frac{3}{2p})} \|\nabla d_{\frac{3}{4}}\|_{L^{\frac{9}{5}}}^2 d\tau \\ & \lesssim \sup_{\tau \in [0, T]} \|\omega(\tau)\|_{L^{\frac{3}{2}}}^{p(1-\frac{3}{2p})} \int_0^T \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 d\tau + \sup_{\tau \in [0, T]} \|d(\tau)\|_{L^{\frac{3}{2}}}^{p(1-\frac{3}{2p})} \int_0^T \|\nabla d_{\frac{3}{4}}\|_{L^2}^2 d\tau \end{aligned} \quad (89)$$

by (11), continuity of Riesz transform in $L^p, p \in (1, \infty)$ and (9) while

$$\begin{aligned} & \int_0^T \|\nabla_h \nabla_h \Delta_h^{-1} \partial_3 u^3\|_{\mathcal{B}_p}^p + \|\nabla_h \nabla_h \Delta_h^{-1} \partial_3 b^3\|_{\mathcal{B}_p}^p d\tau \\ & \lesssim \int_0^T \left| \sup_{j \in \mathbb{Z}} 2^{j(\frac{1}{2} + \frac{2}{p})} \|\dot{\Delta}_j u^3\|_{L^2} \right|^p + \left| \sup_{j \in \mathbb{Z}} 2^{j(\frac{1}{2} + \frac{2}{p})} \|\dot{\Delta}_j b^3\|_{L^2} \right|^p d\tau \\ & \lesssim \int_0^T \|u^3\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}}^p + \|b^3\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}}^p d\tau \end{aligned} \quad (90)$$

by Bernstein's inequality and continuity of Riesz transform in $L^p, p \in (1, \infty)$. Therefore, applying (89) and (90) in (88), by (59) we obtain

$$\int_0^{T^*} \|\nabla_h u^h\|_{\mathcal{B}_p}^p + \|\nabla_h b^h\|_{\mathcal{B}_p}^p d\tau \lesssim 1.$$

Finally, by (5) for any $T < T^*$

$$\begin{aligned} & \int_0^T \|\partial_3 u^h\|_{\mathcal{B}_2}^2 + \|\partial_3 b^h\|_{\mathcal{B}_2}^2 d\tau \\ & \lesssim \int_0^T \|\partial_3 \nabla_h^\perp \Delta_h^{-1} \omega\|_{\mathcal{B}_2}^2 + \|\partial_3 \nabla_h^\perp \Delta_h^{-1} d\|_{\mathcal{B}_2}^2 + \|\partial_3 \nabla_h \Delta_h^{-1} \partial_3 u^3\|_{\mathcal{B}_2}^2 + \|\partial_3 \nabla_h \Delta_h^{-1} \partial_3 b^3\|_{\mathcal{B}_2}^2 d\tau \end{aligned} \quad (91)$$

where

$$\begin{aligned}
& \int_0^T \|\partial_3 \nabla_h^\perp \Delta_h^{-1} \omega\|_{\mathcal{B}_2}^2 + \|\partial_3 \nabla_h^\perp \Delta_h^{-1} d\|_{\mathcal{B}_2}^2 d\tau \\
& \lesssim \int_0^T \left| \sup_{j \in \mathbb{Z}} 2^{-j} \sum_{k \leq j+\delta, l \leq j+\delta} \|\dot{\Delta}_j \dot{\Delta}_k^h \dot{\Delta}_l^v \partial_3 \nabla_h^\perp \Delta_h^{-1} \omega\|_{L^\infty} \right|^2 \\
& \quad + \left| \sup_{j \in \mathbb{Z}} 2^{-j} \sum_{k \leq j+\delta, l \leq j+\delta} \|\dot{\Delta}_j \dot{\Delta}_k^h \dot{\Delta}_l^v \partial_3 \nabla_h^\perp \Delta_h^{-1} d\|_{L^\infty} \right|^2 d\tau \\
& \lesssim \int_0^T \left| \sup_{j \in \mathbb{Z}} 2^{-j} \sum_{k \leq j+\delta, l \leq j+\delta} 2^{l(\frac{2}{3})} 2^{\frac{k}{3}} \|\dot{\Delta}_j \dot{\Delta}_k^h \dot{\Delta}_l^v \partial_3 \omega\|_{L^{\frac{3}{2}}} \right|^2 \\
& \quad + \left| \sup_{j \in \mathbb{Z}} 2^{-j} \sum_{k \leq j+\delta, l \leq j+\delta} 2^{l(\frac{2}{3})} 2^{\frac{k}{3}} \|\dot{\Delta}_j \dot{\Delta}_k^h \dot{\Delta}_l^v \partial_3 d\|_{L^{\frac{3}{2}}} \right|^2 d\tau \\
& \lesssim \int_0^T \|\partial_3 \omega\|_{L^{\frac{3}{2}}}^2 + \|\partial_3 d\|_{L^{\frac{3}{2}}}^2 d\tau \\
& \lesssim \sup_{\tau \in [0, T]} \|\omega_{\frac{3}{4}}(\tau)\|_{L^2}^{\frac{2}{3}} \int_0^T \|\nabla \omega_{\frac{3}{4}}\|_{L^2}^2 d\tau + \sup_{\tau \in [0, T]} \|d_{\frac{3}{4}}(\tau)\|_{L^2}^{\frac{2}{3}} \int_0^T \|\nabla d_{\frac{3}{4}}\|_{L^2}^2 d\tau
\end{aligned} \tag{92}$$

by Bernstein's inequality, and $L^{\frac{3}{2}}(\mathbb{R}^3) \subset \dot{B}_{\frac{3}{2}, 2}^0$ (cf. [5]) and (9) whereas for fixed $\theta \in (\frac{1}{2} - \frac{2}{p}, \frac{1}{6})$

$$\begin{aligned}
& \int_0^T \|\partial_3 \nabla_h \Delta_h^{-1} \partial_3 u^3\|_{\mathcal{B}_2}^2 + \|\partial_3 \nabla_h \Delta_h^{-1} \partial_3 b^3\|_{\mathcal{B}_2}^2 d\tau \\
& \lesssim \int_0^T \left| \sup_{j \in \mathbb{Z}} 2^{-j} \sum_{k \leq j+\delta, l \leq j+\delta} \|\dot{\Delta}_j \dot{\Delta}_k^h \dot{\Delta}_l^v \partial_3^2 \nabla_h \Delta_h^{-1} u^3\|_{L^\infty} \right|^2 \\
& \quad + \left| \sup_{j \in \mathbb{Z}} 2^{-j} \sum_{k \leq j+\delta, l \leq j+\delta} \|\dot{\Delta}_j \dot{\Delta}_k^h \dot{\Delta}_l^v \partial_3^2 \nabla_h \Delta_h^{-1} b^3\|_{L^\infty} \right|^2 d\tau \\
& \lesssim \int_0^T \left| \sup_{j \in \mathbb{Z}} 2^{-j} \sum_{k \leq j+\delta, l \leq j+\delta} 2^{k(\frac{1}{2}-\theta)} 2^{l(\frac{1}{2}+\theta)} 2^{k(\theta-\frac{1}{2})} 2^{-l\theta} \|\dot{\Delta}_j \dot{\Delta}_k^h \dot{\Delta}_l^v \partial_3^2 u^3\|_{L^2} \right|^2 \\
& \quad + \left| \sup_{j \in \mathbb{Z}} 2^{-j} \sum_{k \leq j+\delta, l \leq j+\delta} 2^{k(\frac{1}{2}-\theta)} 2^{l(\frac{1}{2}+\theta)} 2^{k(\theta-\frac{1}{2})} 2^{-l\theta} \|\dot{\Delta}_j \dot{\Delta}_k^h \dot{\Delta}_l^v \partial_3^2 b^3\|_{L^2} \right|^2 d\tau \\
& \lesssim \int_0^T \|\partial_3^2 u^3\|_{\mathcal{H}_\theta}^2 + \|\partial_3^2 b^3\|_{\mathcal{H}_\theta}^2 d\tau
\end{aligned} \tag{93}$$

by Bernstein's inequality. Thus, applying (92) and (93) in (91), by (59) and (60),

$$\int_0^{T^*} \|\partial_3 u^h\|_{\mathcal{B}_2}^2 + \|\partial_3 b^h\|_{\mathcal{B}_2}^2 d\tau \lesssim 1.$$

Due to Proposition 4.1, this completes the proof of (4).

6. APPENDIX

6.1. Local theory of Theorem 1.1. We let $X^\pm \triangleq u \pm b, Y^\pm \triangleq \Omega \pm j$ so that from (18a), (18b) and (19)

$$\partial_t Y^\pm - \Delta Y^\pm + (X^\mp \cdot \nabla) Y^\pm = (Y^\pm \cdot \nabla) X^\mp \pm 2M(u, b).$$

By Sobolev embedding of $\dot{W}^{\frac{1}{2}, \frac{3}{2}}(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$, and continuity of Riesz transform in $L^{\frac{3}{2}}(\mathbb{R}^3)$, we have $u_0, b_0 \in \dot{B}_{2,2}^{\frac{1}{2}}(\mathbb{R}^3)$. By [23] (also [35]), we find a unique solution pair $u, b \in C(0, T^*; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)) \cap L_{loc}^2(0, T^*; \dot{H}^{\frac{3}{2}}(\mathbb{R}^3))$. Now by Lemma 3.1 [12],

$$\begin{aligned} & \frac{2}{3} \|Y^+\|_{L^{\frac{3}{2}}}^{\frac{3}{2}} + \frac{1}{2} \int_0^t |\nabla Y^+|^2 |Y^+|^{-\frac{1}{2}} d\tau \\ &= \frac{2}{3} \|Y_0^+\|_{L^{\frac{3}{2}}}^{\frac{3}{2}} + \int_0^t \int [(Y^+ \cdot \nabla) X^- + 2M(u, b)] |Y^+|^{-\frac{1}{2}} Y^+ d\tau \\ &\leq \frac{2}{3} \|Y_0^+\|_{L^{\frac{3}{2}}}^{\frac{3}{2}} + \int_0^t \|Y^+\|_{L^3} \|\nabla X^-\|_{L^3} \|Y^+\|_{L^{\frac{3}{2}}}^{\frac{1}{2}} + \|\nabla u\|_{L^3} \|\nabla b\|_{L^3} \|Y^+\|_{L^{\frac{3}{2}}}^{\frac{1}{2}} d\tau \\ &\lesssim \|Y_0^+\|_{L^{\frac{3}{2}}}^{\frac{3}{2}} + \int_0^t \|u\|_{\dot{H}^{\frac{3}{2}}}^2 + \|b\|_{\dot{H}^{\frac{3}{2}}}^2 d\tau + \int_0^t (\|u\|_{\dot{H}^{\frac{3}{2}}}^2 + \|b\|_{\dot{H}^{\frac{3}{2}}}^2) \|Y^+\|_{L^{\frac{3}{2}}}^{\frac{3}{2}} d\tau \end{aligned}$$

by Hölder's inequalities, Sobolev embedding of $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$ and Young's inequality so that for $T < T^*$

$$\sup_{t \in [0, T]} \|Y^+(t)\|_{L^{\frac{3}{2}}}^{\frac{3}{2}} + \int_0^T |\nabla Y^+|^2 |Y^+|^{-\frac{1}{2}} d\tau \lesssim (e + \|Y_0^+\|_{L^{\frac{3}{2}}}^{\frac{3}{2}}) e^{\int_0^T \|u\|_{\dot{H}^{\frac{3}{2}}}^2 + \|b\|_{\dot{H}^{\frac{3}{2}}}^2 d\tau}.$$

Similar procedure on the equation of Y^- gives in sum

$$\begin{aligned} & \sup_{t \in [0, T]} (\|\Omega\|_{L^{\frac{3}{2}}}^{\frac{3}{2}} + \|j\|_{L^{\frac{3}{2}}}^{\frac{3}{2}})(t) + \int_0^T |\nabla(\Omega + j)|^2 |\Omega + j|^{-\frac{1}{2}} + |\nabla(\Omega - j)|^2 |\Omega - j|^{-\frac{1}{2}} d\tau \\ & \lesssim (1 + \|\Omega_0\|_{L^{\frac{3}{2}}}^{\frac{3}{2}} + \|j_0\|_{L^{\frac{3}{2}}}^{\frac{3}{2}}) \exp \left(\int_0^T \|u\|_{\dot{H}^{\frac{3}{2}}}^2 + \|b\|_{\dot{H}^{\frac{3}{2}}}^2 d\tau \right) \lesssim 1. \end{aligned}$$

This completes the proof of the local theory of Theorem 1.1.

6.2. Additional estimates. Here we prove two additional estimates:

$$\|fg\|_{\dot{H}^{s-1, \sigma_1+\sigma_2-\frac{1}{2}}} \lesssim \|f\|_{(\dot{B}_{2,1}^1)_h (\dot{B}_{2,1}^{\sigma_1})_v} \|g\|_{\dot{H}^{s-1, \sigma_2}}, \quad (94)$$

where $\sigma_1 < \frac{1}{2}, \sigma_2 < \frac{1}{2}, \sigma_1 + \sigma_2 > 0, 2 > s > 0$ and

$$\|fg\|_{\dot{H}^{s_1+s_2-1, \sigma-\frac{1}{2}}} \lesssim \|f\|_{(\dot{B}_{2,2}^{s_1})_h (\dot{B}_{2,1}^{\frac{1}{2}})_v} \|g\|_{\dot{H}^{s_2, \sigma-\frac{1}{2}}} \quad (95)$$

where $s_1 < 1, s_2 < 1, s_1 + s_2 > 0, 1 > \sigma > 0$. Since these are standard applications, we only sketch (94); the proof of (95) is similar.

Due to the following horizontal and vertical Bony paraproduct decompositions

$$\begin{aligned} T^h(f, g) &\triangleq \sum_m \dot{S}_{m-1}^h f \dot{\Delta}_m^h g, \quad R^h(f, g) \triangleq \sum_m \dot{\Delta}_m^h f \tilde{\Delta}_m^h g, \quad \tilde{T}^h(f, g) \triangleq T^h(g, f), \\ T^v(f, g) &\triangleq \sum_n \dot{S}_{n-1}^v f \dot{\Delta}_n^v g, \quad R^v(f, g) \triangleq \sum_n \dot{\Delta}_n^v f \tilde{\Delta}_n^v g, \quad \tilde{T}^v(f, g) \triangleq T^v(g, f), \end{aligned}$$

we can write $fg = (T^h + R^h + \tilde{T}^h)(T^v + R^v + \tilde{T}^v)(f, g)$ in nine parts: e.g.

$$R^h T^v(f, g) = \sum_{m,n} \dot{\Delta}_m^h \dot{S}_{n-1}^v f \tilde{\Delta}_m^h \dot{\Delta}_n^v g.$$

Let us estimate this term:

$$\begin{aligned}
& \|\dot{\Delta}_k^h \dot{\Delta}_l^v \sum_{k \leq m+\delta, |l-n| \leq 4} \dot{\Delta}_m^h \dot{S}_{n-1}^v f \tilde{\Delta}_m^h \dot{\Delta}_n^v g\|_{L^2} \\
& \lesssim 2^k \sum_{k \leq m+\delta} \|\dot{\Delta}_m^h \dot{S}_{l-1}^v f \tilde{\Delta}_m^h \dot{\Delta}_l^v g\|_{L_h^1 L_v^2} \\
& \lesssim 2^k \sum_{k \leq m+\delta} \sum_{l' \leq l-2} \|\dot{\Delta}_m^h \dot{\Delta}_{l'}^v f\|_{L_h^2 L_v^\infty} \|\tilde{\Delta}_m^h \dot{\Delta}_l^v g\|_{L^2} \\
& \lesssim 2^k \sum_{k \leq m+\delta} \sum_{l' \leq l-2} 2^{\frac{l'}{2}} \|\dot{\Delta}_m^h \dot{\Delta}_{l'}^v f\|_{L^2} \|\tilde{\Delta}_m^h \dot{\Delta}_l^v g\|_{L^2}
\end{aligned}$$

by Bernstein's and Hölder's inequalities. Thus,

$$\begin{aligned}
& 2^{k(s-1)} 2^{l(\sigma_1+\sigma_2-\frac{1}{2})} \|\dot{\Delta}_k^h \dot{\Delta}_l^v R^h T^v(f, g)\|_{L^2} \\
& \lesssim \sum_{\alpha=-1}^1 \sum_{k \leq m+\delta} \sum_{l' \leq l-2} 2^{(k-m)s} 2^{(l'-l)(\frac{1}{2}-\sigma_1)} 2^m 2^{l'\sigma_1} \|\dot{\Delta}_m^h \dot{\Delta}_{l'}^v f\|_{L^2} \\
& \quad \times 2^{\alpha(s-1)} 2^{(m-\alpha)(s-1)} 2^{l\sigma_2} \|\dot{\Delta}_{m-\alpha}^h \dot{\Delta}_l^v g\|_{L^2}.
\end{aligned}$$

We take l^2 -norm in k now to obtain

$$\begin{aligned}
& \left\| \left(2^{k(s-1)} 2^{l(\sigma_1+\sigma_2-\frac{1}{2})} \|\dot{\Delta}_k^h \dot{\Delta}_l^v R^h T^v(f, g)\|_{L^2} \right)_k \right\|_{l^2} \\
& \lesssim \sum_{\alpha=-1}^1 \left\| \left(\sum_{l' \leq l-2} 2^{(l'-l)(\frac{1}{2}-\sigma_1)} 2^k 2^{l'\sigma_1} \|\dot{\Delta}_k^h \dot{\Delta}_{l'}^v f\|_{L^2} 2^{(k-\alpha)(s-1)} 2^{l\sigma_2} \|\dot{\Delta}_{k-\alpha}^h \dot{\Delta}_l^v g\|_{L^2} \right)_k \right\|_{l^1}
\end{aligned}$$

by Young's inequality for convolution. We now take l^2 -norm in l and use Minkowski's inequality to obtain

$$\begin{aligned}
& \left\| \left(2^{k(s-1)} 2^{l(\sigma_1+\sigma_2-\frac{1}{2})} \|\dot{\Delta}_k^h \dot{\Delta}_l^v R^h T^v(f, g)\|_{L^2} \right)_{k,l} \right\|_{l^2} \\
& \lesssim \sum_{\alpha=-1}^1 \left\| \left(\left\| \left(\sum_{l' \leq l-2} 2^{(l'-l)(\frac{1}{2}-\sigma_1)} 2^k 2^{l'\sigma_1} \|\dot{\Delta}_k^h \dot{\Delta}_{l'}^v f\|_{L^2} 2^{(k-\alpha)(s-1)} 2^{l\sigma_2} \|\dot{\Delta}_{k-\alpha}^h \dot{\Delta}_l^v g\|_{L^2} \right)_k \right\|_{l^2} \right)_l \right\|_{l^1} \\
& \lesssim \sum_{\alpha=-1}^1 \left\| \left(\left\| \left(2^k 2^{l\sigma_1} \|\dot{\Delta}_k^h \dot{\Delta}_l^v f\|_{L^2} \right)_l \right\|_{l^2} \left\| \left(2^{(k-\alpha)(s-1)} 2^{l\sigma_2} \|\dot{\Delta}_{k-\alpha}^h \dot{\Delta}_l^v g\|_{L^2} \right)_l \right\|_{l^2} \right)_k \right\|_{l^1} \\
& \lesssim \|f\|_{(\dot{B}_{2,1}^1)_h} \|\dot{B}_{2,1}^{\sigma_1}\|_v \|g\|_{\dot{H}^{s-1, \sigma_2}}
\end{aligned}$$

by Hölder's inequality and Young's inequality for convolution.

The other terms are similar and we refer to e.g. [19, 29] for details.

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